A folk model structure on omega-cat

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Abstract

The primary aim of this work is an intrinsic homotopy theory of strict ω-categories. We establish a model structure on ω**Cat**, the category of strict ω-categories. The constructions leading to the model structure in question are expressed entirely within the scope of ω**Cat**, building on a set of generating cofibrations and a class of weak equivalences as basic items. All object are fibrant while free objects are cofibrant. We further exhibit model structures of this type on n-categories for arbitrary $n \in \mathbb{N}$, as specialisations of the ω-categorical one along right adjoints. In particular, known cases for $n = 1$ and $n = 2$ nicely fit into the scheme.

1 Introduction

1.1 Background and motivations

The origin of the present work goes back to the following result [1, 24]:

> *if a monoid M can be presented by a finite, confluent and terminating rewriting system, then its third homology group $\mathrm{H}_3(M)$ is of finite type.*

The finiteness property extends in fact to all dimensions [14], but the above theorem may also be refined in another direction: the same hypothesis implies that M has *finite derivation type* [25], a property of homotopical nature.

We claim that these ideas are better expressed in terms of ω-categories (see [10, 11, 17]). Thus we work in the category ω**Cat**, whose objects are the strict ω-categories and the morphisms are ω-functors (see Section 3). In fact, when considering the interplay between the monoid itself and the space of computations attached to any presentation of it, one readily observes that both objects support a structure of ω-category in a very direct way: this was the starting point of [19], which introduces a notion of *resolution* for ω-categories, based on computads [26, 21] or polygraphs [6], the terminology we adopt here. Recall that a polygraph S consists of sets of cells of all dimensions, determining a freely generated ω-category S^*. A resolution of an ω-category C by a polygraph S is then an ω-functor $p : S^* \to C$ satisfying a certain lifting property (see Section 5 below); [19] also defines a homotopy relation between ω-functors and shows that any two resolutions of the same ω-category are homotopically equivalent in this sense.

This immediately suggests looking for a homotopy theory on ω**Cat** in which the above resolutions become trivial fibrations: the model structure we describe here does exactly that. Notice, in addition, that polygraphs turn out to be the cofibrant objects (see [20] and Section 5 below). On the other hand, our model structure generalizes in a very precise sense the "folk" model structure on **Cat** (see [13]) as well a model structure on 2**Cat** in a similar spirit (see [15, 16]). Incidentally, there is also a quite different, Thomason-like, model structure on 2**Cat** (see [27]). Its generalisation to ω**Cat** remains an open problem.

Since [22], the notion of model structure has been gradually recognized as the appropriate abstract framework for developing homotopy theory in a category **C**: it consists in three classes of morphisms, *weak equivalences*, *fibrations*, and *cofibrations*, subject to axioms whose exact formulation has somewhat evolved in time. In practice, most model structures are *cofibrantly generated*. This means that there are sets I of *generating cofibrations* and J of *generating trivial cofibrations* which determine all the cofibrations and all the fibrations by lifting properties.

Recall that, given a set I of morphisms, I-*injectives* are the morphisms which have the *right lifting property* with respect to I. They build a class denoted by $I-\mathrm{inj}$. Likewise, I-*cofibrations* are the morphisms having the *left lifting property* with respect to $I-\mathrm{inj}$ (see Section 2.2). The class of I-cofibrations is denoted by $I-\mathrm{cof}$. Now, our

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construction is based on a theorem by J.Smith (see [3]): under some fairly standard assumptions on the underlying category, conditions

(S1) *$\mathcal{W}$ has the 3 for 2 property and is stable under retracts;*

(S2) $I{-}\mathrm{inj} \subseteq \mathcal{W}$*;*

(S3) $I{-}\mathrm{cof} \cap \mathcal{W}$ *is closed under pushouts and transfinite compositions;*

(S4) *$\mathcal{W}$ admits a solution set $J \subseteq I{-}\mathrm{cof} \cap \mathcal{W}$ at I.*

are sufficient to obtain a model structure in which $\mathcal{W}$, I and J are the weak equivalences, the generating cofibrations and the generating trivial cofibrations, respectively.

1.2 Organization of the paper

Section 2 reviews *combinatorial model categories*, with special emphasis on our version of Smith's theorem (Section 2.4), while Section 3 recalls the basic definitions of globular sets and ω-categories, and sets the notations.
Section 4 is the core of the paper, that is the derivation of our model structure by means of a set I of generating cofibrations and a class $\mathcal{W}$ of weak equivalences, satisfying conditions **(S1)** to **(S4)** .

1.2.1 Sketch of the main argument

We first define the set I of generating cofibrations, and establish closure properties we shall use later in the proof of condition **(S3)** .
We then define the class $\mathcal{W}$ of ω-weak equivalences, which are at this stage our candidates for the rôle of weak equivalences (Section 4.3). For this purpose, we first need a notion of ω-equivalence between parallel cells (Section 4.2), together with crucial properties of this notion.
We then prove condition **(S2)** , and part of **(S1)** (Section 4.3), as well as additional closure properties contributing to **(S3)** .
At this stage, just one point of **(S1)** remains unproved, namely the assertion

if $f : X \to Y$ and $g \circ f : X \to Z$ belong to $\mathcal{W}$, then so does $g : Y \to Z$.

This requires an entirely new construction: we define an endofunctor Γ of $\omega\mathbf{Cat}$, which to each ω-category X associates an ω-category $\Gamma(X)$ of *reversible cylinders* in X. Section 4.4 summarizes the main features of Γ, whereas the more technical proofs are given in Appendix A. This eventually leads to an alternative characterization of weak equivalences and to a complete proof of **(S1)** .
As for condition **(S3)** , the difficult point is to prove the closure of $I{-}\mathrm{cof} \cap \mathcal{W}$ by pushout, which does not follow from the previously established properties. The main obstacle is that $\mathcal{W}$ itself is definitely *not* closed by pushout. What we need instead is a new class $\mathcal{Z}$ of *immersions* such that:

i. $\mathcal{Z}$ is closed by pushout;

ii. $I{-}\mathrm{cof} \cap \mathcal{W} \subseteq \mathcal{Z} \subseteq \mathcal{W}$,

which completes the proof of **(S3)** . Immersions are defined in Section 4.6, by using again the functor Γ in an essential way.
Section 4.7 is devoted to the proof of the *solution set condition* **(S4)** . Precisely, we have to build, for each $i \in I$, a *set* J_i of ω-functors satisfying the following property: for each commutative square

$$\begin{tikzcd} X \arrow[r] \arrow[d, "i"'] & Z \arrow[d, "f"] \\ Y \arrow[r] & T \end{tikzcd} \tag{1}$$

where $i \in I$ and $f \in \mathcal{W}$, there is a $j \in J_i$ such that (1) factors through j:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "i"'] & U \arrow[r] \arrow[d, "j"'] & Z \arrow[d, "f"] \\ Y \arrow[r] & V \arrow[r] & T \end{tikzcd} \tag{2}$$

The whole solution set is then $J \overset{\mathrm{def.}}{=} \bigcup_{i \in I} J_i$. It turns out that in our case, the sets J_i are just singletons.

1.2.2 Additional properties

The end of the paper is devoted to two additional points: Section 5 gives a characterization of cofibrant objects as polygraphs by interpreting the results of [20] in terms of our model structure. Finally, Section 6 shows how the present model structure on ω**Cat** transfers to n**Cat** for any integer n: in particular, for $n = 1$ and $n = 2$, we recover the abovementioned structures on **Cat** [13] and 2**Cat** [15, 16].

2 Combinatorial model categories

We recall some facts about model categories with locally-presentable underlying categories.

2.1 Locally presentable categories

Let α be a regular cardinal. An α-filtered category **F** is a category such that

i. for any set of objects S with cardinality $|S| < \alpha$ and for each $A \in S$ there is an object T and a morphism $f_A : A \to T$;

ii. for any set of morphisms $M \subseteq \mathbf{F}(A, B)$ with cardinality $|M| < \alpha$ there is an object C and a morphism $m : B \to C$, such that $m \circ m' = m \circ m''$ for all $m', m'' \in M$.

We say that **F** is filtered in case $\alpha = \aleph_0$. In particular, a *directed* (partially ordered) set is a filtered category.
Recall that an α-filtered colimit is a colimit of a functor $D : \mathbf{I} \to \mathbf{C}$ from a small α-filtered category **I**. Let **C** be a category. An object $X \in \mathbf{C}$ is α-presentable if the covariant representable functor $\mathbf{C}(X, -) : \mathbf{C} \to \mathbf{Sets}$ preserves α-filtered colimits. This boils down to the fact that a morphism from X to an α-filtered colimit factors through some object of the relevant α-filtered diagram, in an essentially unique way. If X is α-presentable, and β is a regular cardinal such that $\alpha < \beta$, then X is also β-presentable.
We say that an object $X \in \mathbf{C}$ is *presentable* if there is a regular cardinal witnessing this fact. If it is the case, the smallest such cardinal, $\pi(X)$, is called X's *presentation rank*.

Definition 1. *Let α be a regular cardinal. A cocomplete category* **C** *is* locally α-presentable *if there is a family $G = (G_i)_{i\in I}$ of objects such that every object of* **C** *is an α-filtered colimit of a diagram in the full subcategory spanned by the G_i's. We say that a cocomplete category is* locally finitely presentable *if it is locally $\aleph_0$-presentable. Finally, we say that a cocomplete category is* locally presentable *if there is a regular cardinal witnessing this fact.*

Definition 1 is equivalent to the original one by Gabriel and Ulmer [7]. It proves especially powerful to establish factorisation results, when combined with the *small object argument* (Section 2.2). Let β be a regular cardinal. Recall that a β-colimit is a colimit of a functor $D : \mathbf{I} \to \mathbf{C}$ from a small category **I** such that $|\mathbf{I}_1| < \beta$.

Proposition 1. *Let β be a regular cardinal. A β-colimit of β-presentable objects is β-presentable.*

Remark 1. Let α be a regular cardinal and **C** be a locally α-presentable category. By definition of local presentability, every object $X \in \mathbf{C}$ is an α-filtered colimit of a diagram of α-presentable objects, so it is a β-colimit for a regular cardinal β such that $\alpha \leqslant \beta \leqslant |\mathbf{C}_1|^+$. Thus, by virtue of Proposition 1, every object of **C** is presentable (with a presentation rank possibly exceeding α). $\Diamond$

2.2 Small objects for free

Let **C** be a category. Recall that its *category of morphisms* $\mathbf{C}^{\to}$ is defined as the functor category $\mathbf{C}^{(\cdot\to\cdot)}$, where $(\cdot \to \cdot)$ is the category generated by the one-arrow graph. Let $f : X \to Y$ and $g : Z \to T$ be morphisms in **C**. We say that f has the *left-lifting* property with respect to g, or equivalently that g has the *right lifting* property with respect to f, if every commuting square $(u, v) \in \mathbf{C}^{\to}(f, g)$ admits a *lift*, that is a morphism $h : Y \to T$ making the following diagram commutative

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Z \arrow[d, "g"] \\ Y \arrow[r, "v"'] \arrow[ur, dotted, "h"] & T \end{tikzcd}$$

This relation is denoted by $f \pitchfork g$.
For any class of morphisms $\mathcal{A}$, we define

$$
\begin{aligned}
{}^{\pitchfork}\mathcal{A} &\stackrel{\text{def.}}{=} \{f \mid f \pitchfork g, g \in \mathcal{A}\} \\
\mathcal{A}^{\pitchfork} &\stackrel{\text{def.}}{=} \{g \mid f \pitchfork g, f \in \mathcal{A}\}
\end{aligned}
$$

Proposition 2. *Suppose $f = f'' \circ f'$. Then*

- *if $f' \pitchfork f$ then f is a retract of f'';*
- *if $f \pitchfork f''$ then f is a retract of f'.*

Proposition 2 is known as "the retract argument".
Let $\mathrm{dom} : \mathbf{C}^{\rightarrow} \to \mathbf{C}$ and $\mathrm{cod} : \mathbf{C}^{\rightarrow} \to \mathbf{C}$ be the obvious functors picking the domain and the codomain of a morphism, respectively. A *functorial factorisation* in $\mathbf{C}$ is a triple

$$F = (F, \lambda, \rho)$$

where $F : \mathbf{C}^{\rightarrow} \to \mathbf{C}$ is a functor while $\lambda : \mathrm{dom} \to F$ and $\rho : F \to \mathrm{cod}$ are natural transformations. Let $\mathcal{L}$ and $\mathcal{R}$ be classes of morphisms in $\mathbf{C}$. We say that the pair $(\mathcal{L}, \mathcal{R})$ admits a functorial factorisation (F, λ, ρ) provided that $\lambda_f \in \mathcal{L}$ and $\rho_f \in \mathcal{R}$ for all morphisms $f \in \mathbf{C}^{\rightarrow}$. If (F, λ, ρ) is clear from the context (or if it does not matter), we say by abuse of language that $(\mathcal{L}, \mathcal{R})$ *is* a functorial factorisation.
Let I be a set of morphisms in a cocomplete category $\mathbf{C}$ and I^* be the closure of I under pushout. The class I-cell of *relative I-cell complexes* is the closure of I^* under transfinite composition. Let $I{-}\mathrm{inj} \stackrel{\text{def.}}{=} I^{\pitchfork}$ and $I{-}\mathrm{cof} \stackrel{\text{def.}}{=} {}^{\pitchfork}(I{-}\mathrm{inj})$.

Remark 2. If $I \subseteq I'$, then $I{-}\mathrm{inj} \supseteq I'{-}\mathrm{inj}$ and $J{-}\mathrm{inj} = (J{-}\mathrm{cof}){-}\mathrm{inj}$. It is easy to see that $I{-}\mathrm{cell} \subseteq I{-}\mathrm{cof}$. $\Diamond$

The next proposition recalls standard formal properties of the classes just defined (see [8]).

Proposition 3. *$I{-}\mathrm{inj}$ as well as $I{-}\mathrm{cof}$ contain all identities. $I{-}\mathrm{inj}$ is closed under composition and pullback while $I{-}\mathrm{cof}$ is closed under retract, transfinite composition and pushout.*

We may now state the crucial factorisation result we shall need:

Proposition 4. *Suppose that $\mathbf{C}$ is locally presentable and let I be a set of morphisms of $\mathbf{C}$. Then $(I{-}\mathrm{cell}, I{-}\mathrm{inj})$ is a functorial factorisation.*

Proof. The required factorisation is produced by the "small object argument", due to Quillen (see also [9] for an extensive discussion):

- For any f in $\mathbf{C}^{\rightarrow}$, let S_f be the set of morphisms of $\mathbf{C}^{\rightarrow}$ with domain in I and codomain f, that is

  $$S_f = \{s = (u_s, v_s) \in \mathbf{C}^{\rightarrow} \mid \mathrm{dom}(s) = i_s \in I,\ \mathrm{cod}(s) = f\}.$$

  We get a functor $F : \mathbf{C}^{\rightarrow} \to \mathbf{C}$ together with natural transformations $\lambda : \mathrm{dom} \to F$ and $\rho : F \to \mathrm{cod}$ determined by the inscribed pushout of the outer commutative square

$$
\begin{tikzcd}
\coprod_{s\in S_f} A_s \arrow[rr, "{[(u_s)_{s\in S_f}]}"] \arrow[dd, "\coprod_{s\in S_f} i_s"'] & & X \arrow[dl, "\lambda_f"'] \arrow[dd, "f"] \\
& F(f) \arrow[dr, dotted, "\rho_f"] \arrow[ul, phantom, "\ulcorner", very near start] & \\
\coprod_{s\in S_f} B_s \arrow[ur, "j_0"] \arrow[rr, "{[(v_s)_{s\in S_f}]}"'] & & Y
\end{tikzcd}
$$

  where, for each $s \in S_f$, $i_s : A_s \to B_s$, and $[(u_s)_{s\in S_f}]$, $[(v_s)_{s\in S_f}]$ are given by the universal property of coproducts.

– By transfinite iteration of the previous construction, we get, for each ordinal β, a triple $(F^\beta, \lambda^\beta, \rho^\beta)$. Precisely,

$$
\begin{aligned}
F^0(f) &\stackrel{\text{def.}}{=} F(f),\\
\lambda^0_f &\stackrel{\text{def.}}{=} \lambda_f,\\
\rho^0_f &\stackrel{\text{def.}}{=} \rho_f;
\end{aligned}
$$

if $\beta+1$ is a successor ordinal, then

$$
\begin{aligned}
F^{\beta+1}(f) &\stackrel{\text{def.}}{=} F\left(\rho^\beta_f\right),\\
\lambda^{\beta+1}_f &\stackrel{\text{def.}}{=} \lambda_{\rho^\beta_f} \circ \lambda^\beta_f,\\
\rho^{\beta+1}_f &\stackrel{\text{def.}}{=} \rho_{\rho^\beta_f},
\end{aligned}
$$

and if β be a limit ordinal, then

$$F^\beta(f) \stackrel{\text{def.}}{=} \operatorname{colim}_{\gamma<\beta} F^\gamma(f)$$

while λ^β_f and ρ^β_f are given by transfinite composition and universal property, respectively.

– Now notice that, for each ordinal β, λ^β_f belongs to $I-\text{cell}$, and that $(\lambda^\beta_f, \rho^\beta_f)$ is a functorial factorisation. It remains to show that there is an ordinal κ for which ρ^κ_f belongs to $I-\text{inj}$. This is where local presentability helps: thus, let κ be a regular cardinal such that for each $i \in I$, the presentation rank $\pi(\operatorname{dom} i)$ is strictly smaller than κ, and suppose that the outer square of the following diagram commutes:

$$
\begin{tikzcd}
A \arrow[rrr, "u"] \arrow[ddd, "i"'] & & & F^\kappa(f) \arrow[ddd, "\rho^\kappa_f"] \\
& & F^{\beta+1}(f) \arrow[ur, dotted, "c^{\beta+1,\kappa}"] & \\
& \coprod_{s\in S_{\rho^\beta_f}} B_s \arrow[ur, dotted, "j_{\beta+1}"] & & \\
B \arrow[ur, dotted, "in_B"'] \arrow[rrr, "v"'] & & & Y
\end{tikzcd}
$$

Since A is κ-presentable and $F^\kappa(f)$ is a κ-filtered colimit, there is a $\beta < \kappa$ such that u factors through $F^\beta(f)$ as $u = c_{\beta,\kappa} \circ u'$ for some u', with $c_{\beta,\kappa} : F^\beta(f) \to F^\kappa(f)$ the colimiting morphism. It follows then from the above construction that $c_{\beta+1,\kappa} \circ j_{\beta+1} \circ \text{in}_B$ is a lift, whence $\rho^\kappa_f \in I-\text{inj}$, and we are done. $\triangleleft$

2.3 Model structures and cofibrant generation

We say that a class $\mathcal{A}$ of morphisms has the 3 *for* 2 *property* if whenever $h = g \circ f$ and any two out of the three morphisms f, g, h belong to $\mathcal{A}$, then so does the third. We now recall the basics of model structures, following the presentation of [12].

Definition 2. *A model structure on a complete and cocomplete category* **C** *is given by three classes of morphisms, the class* $\mathcal{C}$ *of cofibrations, the class* $\mathcal{F}$ *of fibrations, and the class* $\mathcal{W}$ *of weak equivalences, satisfying the following conditions:*

(M1) $\mathcal{W}$ *has the* 3 *for* 2 *property;*

(M2) $\mathcal{C}$, $\mathcal{F}$ *and* $\mathcal{W}$ *are stable under retracts;*

(M3) $\mathcal{C} \cap \mathcal{W} \subseteq {}^{\pitchfork}\mathcal{F}$ *and* $\mathcal{F} \cap \mathcal{W} \subseteq \mathcal{C}^{\pitchfork}$*;*

(M4) *the pairs* $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ *and* $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ *are functorial factorisations.*

A complete and cocomplete category equipped with a model structure is called a model category. The members of $\mathcal{F} \cap \mathcal{W}$ are called trivial fibrations and the members of $\mathcal{C} \cap \mathcal{W}$ are trivial cofibrations.

Remark 3. There is a certain amount of redundancy in the definition of a model category as the class of fibrations is determined by the class of cofibrations and vice-versa: we have

- $\mathcal{F} = (\mathcal{C} \cap \mathcal{W}) - \mathrm{inj}$
- $\mathcal{F} \cap \mathcal{W} = \mathcal{C} - \mathrm{inj}$;

as well as

- $\mathcal{C} = {}^{\pitchfork}(\mathcal{F} \cap \mathcal{W})$;
- $\mathcal{C} \cap \mathcal{W} = {}^{\pitchfork}\mathcal{F}$. ◊

In most known model categories cofibrations and fibrations are generated by *sets* of morphisms. In the case of locally-presentable categories, we get the following definition:

Definition 3. *A locally-presentable model category is cofibrantly generated if there are two sets I, J of morphisms such that*

i. $\mathcal{C} = I - \mathrm{cof}$*;*

ii. $\mathcal{C} \cap \mathcal{W} = J - \mathrm{cof}$.

The morphisms in I are called generating cofibrations *while the morphisms in J are called* generating trivial cofibrations. *Locally-presentable, cofibrantly generated model categories are called* combinatorial model categories.

Notice that a locally-presentable model category is combinatorial if and only if $\mathcal{F} \cap \mathcal{W} = I - \mathrm{inj}$ and $\mathcal{F} = J - \mathrm{inj}$. The whole point in the definition of combinatorial model categories is the possibility to apply the small object argument to arbitrary sets I and J. The general case, however, requires extra conditions on those sets.

2.4 The solution set condition

Let **C** be a category, i a morphism of **C** and $\mathcal{W}$ a class of morphisms of **C**. We say that $\mathcal{W}$ admits a *solution set at* i if there is a *set* W_i of morphisms such that any commutative square

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "i"'] & \bullet \arrow[d, "w \in \mathcal{W}"] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

where $w \in \mathcal{W}$ factors through some $w' \in \mathcal{W}_i$:

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "i"'] & \bullet \arrow[r] \arrow[d, "w' \in W_i"] & \bullet \arrow[d, "w \in \mathcal{W}"] \\ \bullet \arrow[r] & \bullet \arrow[r] & \bullet \end{tikzcd}$$

If I is a set of morphisms, we say that $\mathcal{W}$ admits a *solution set at I* if it admits a solution set at any $i \in I$.
We now turn to Smith's theorem, on which our construction is based:

Theorem 1. *Let I be a set, and $\mathcal{W}$ a class of morphisms in a locally presentable category* **C**. *Suppose that*

(S1) *$\mathcal{W}$ has the 3 for 2 property and is stable under retracts;*

(S2) $I - \mathrm{inj} \subseteq \mathcal{W}$*;*

(S3) $I - \mathrm{cof} \cap \mathcal{W}$ *is closed under pushouts and transfinite compositions;*

(S4) *$\mathcal{W}$ admits a solution set $J \subseteq I - \mathrm{cof} \cap \mathcal{W}$ at I.*

Then **C** *is a combinatorial model category where $\mathcal{W}$ is the class of weak equivalences while I is a set of generating cofibrations and J is a set of generating trivial cofibrations.*

We refer to [3] for an extensive discussion of Theorem 1. In the original statement, **(S4)** only requires the existence of a solution set, without any inclusion condition. The present version brings a minor simplification in the treatment of our particular case.
For the remaining of this section, we assume the hypotheses of Theorem 1.

Lemma 1. **(Smith)** *Suppose there is a class* $\mathcal{J} \subseteq I{-}\mathrm{cof} \cap \mathcal{W}$ *such that each commuting square*

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "i"'] & \bullet \arrow[d, "w\in\mathcal{W}"] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

admits a factorisation

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "i"'] & \bullet \arrow[r] \arrow[d, "j\in\mathcal{J}"] & \bullet \arrow[d, "w\in\mathcal{W}"] \\ \bullet \arrow[r] & \bullet \arrow[r] & \bullet \end{tikzcd}$$

Then

$$\mathcal{J}{-}\mathrm{cof} = I{-}\mathrm{cof} \cap \mathcal{W}$$

Lemma 1 is a key step in the proof of Theorem 1. This is Lemma 1.8 in [3], where a complete proof is given, based again on the small object argument combined with an induction step.

Remark 4. We have

$$J{-}\mathrm{cof} = I{-}\mathrm{cof} \cap \mathcal{W}$$

by Lemma 1, so in particular

$$J{-}\mathrm{inj} = (I{-}\mathrm{cof} \cap \mathcal{W}){-}\mathrm{inj}$$

by remark 2. ◊

Lemma 2. $I{-}\mathrm{inj} = J{-}\mathrm{inj} \cap \mathcal{W}$.

Proof. "⊆" Since $I{-}\mathrm{inj} \subseteq \mathcal{W}$, by **(S2)** , we need to show that $I{-}\mathrm{inj} \subseteq J{-}\mathrm{inj}$. Let $j \in J$, $f \in I{-}\mathrm{inj}$ and suppose $f \circ u = v \circ u$ for some u and v. The small object argument produces a factorisation $f = f'' \circ f'$ with $f' \in J{-}\mathrm{cof}$ and $f'' \in J{-}\mathrm{inj}$, so there are p and q such that the following diagram commutes (the existence of q is a consequence of Remark 4):

$$\begin{tikzcd} \bullet \arrow[r, "u"] \arrow[dd, "J\ni j"'] & \bullet \arrow[r, equal] \arrow[d, "J-cof\ni f'"'] & \bullet \arrow[ddl, "f\in I-inj"] \\ & \bullet \arrow[ur, dotted, "q"] \arrow[d, "J-inj\ni f''"'] & \\ \bullet \arrow[r, "v"'] \arrow[ur, dotted, bend left, "p"] & \bullet & \end{tikzcd}$$

"⊇" Let $f \in J{-}\mathrm{inj} \cap \mathcal{W}$. The small object argument produces a factorisation

$$\begin{tikzcd} & \bullet \arrow[dl, "I-cof\ni f'"'] \arrow[dd, "f\in(J-inj)\cap\mathcal{W}"] \\ \bullet \arrow[dr, "I-inj\ni f''"'] & \\ & \bullet \end{tikzcd}$$

so $f' \in I{-}\mathrm{cof} \cap \mathcal{W}$ by **(S1)** . On the other hand $f \in (I{-}\mathrm{cof} \cap \mathcal{W}){-}\mathrm{inj}$ by Remark 4, so $f \in I{-}\mathrm{inj}$ by the retract argument (see Proposition 2). ◁

Proof of Theorem 1. Let $\mathcal{C} \overset{\text{def.}}{=} I{-}\mathrm{cof}$ and $\mathcal{F} \overset{\text{def.}}{=} J{-}\mathrm{inj}$. It readily follows that $\mathcal{W}$, $\mathcal{C}$ and $\mathcal{F}$ are the constituent classes of a model structure on **C**:

- **(M1)** holds by hypothesis;
- **(M2)** holds by hypothesis for $\mathcal{W}$, by construction for $\mathcal{C}$ and $\mathcal{F}$;
- as for **(M3)** , consider a commutative square:

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "I-cof\ni c"'] & \bullet \arrow[d, "f\in J-inj"] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

  If $c \in \mathcal{W}$ then this square admits a lift by Remark 4. On the other hand, if $f \in \mathcal{W}$ then this square admits a lift by Lemma 2;

- **(M4)** holds because the factorisations are constructed using the small object argument and have the required properties by Lemma 2 and Remark 4, respectively.

Therefore **C** is a combinatorial model category by Remark 4. ◁

3 Higher dimensional categories

This section is devoted to a brief review of higher dimensional categories, here defined as globular sets with structure.

3.1 Globular sets

Let **O** be the small category whose objects are integers $0, 1, \ldots$, and whose morphisms are generated by $\mathrm{s}_n, \mathrm{t}_n : n \to n{+}1$ for $n \in \mathbb{N}$, subject to the following equations:

$$\begin{aligned} \mathrm{s}_{n+1} \circ \mathrm{s}_n &= \mathrm{t}_{n+1} \circ \mathrm{s}_n, \\ \mathrm{s}_{n+1} \circ \mathrm{t}_n &= \mathrm{t}_{n+1} \circ \mathrm{t}_n. \end{aligned}$$

These equations imply that there are exactly two morphisms from m to n if $m < n$, none if $m > n$, and only the identity if $m = n$.

Definition 4. *A* globular set *is a presheaf on* **O**.

In other words, a globular set is a functor from $\mathbf{O}^{\mathrm{op}}$ to **Sets**. Globular sets and natural transformations form a category **Glob**. If X is a globular set, we denote by X_n the image of $n \in \mathbb{N}$ by X; members of X_n are called n*-cells*. By defining $\sigma_n = X(\mathrm{s}_n)$ and $\tau_n = X(\mathrm{t}_n)$, we get *source* and *target* maps

$$\sigma_n, \tau_n : X_{n+1} \to X_n.$$

More generally, whenever $m > n$, one defines

$$\begin{aligned} \sigma_{n,m} &= \sigma_n \circ \cdots \circ \sigma_{m-1}, \\ \tau_{n,m} &= \tau_n \circ \cdots \circ \tau_{m-1}, \end{aligned}$$

so that $\sigma_{n,m}$ and $\tau_{n,m}$ are maps from X_m to X_n. Let us call two n-cells x, y *parallel* whenever $n = 0$, or $n > 0$ and

$$\begin{aligned} \sigma_{n-1}(x) &= \sigma_{n-1}(y), \\ \tau_{n-1}(x) &= \tau_{n-1}(y). \end{aligned}$$

We write $x \parallel y$ whenever x, y, are parallel cells:

$$\begin{tikzcd} \bullet \arrow[r, bend left, "x"] \arrow[r, bend right, "y"'] & \bullet \end{tikzcd}$$

We will need a few additional notations about globular sets:

– if u is an n+1-cell, we write $u : x \to y$ whenever $\sigma_n u = x$ and $\tau_n u = y$, in which case $x \parallel y$;

– if $m > n$ and u is an m-cell, we write $u : x \to_n y$ whenever $\sigma_{n,m}(u) = x$ and $\tau_{n,m}(u) = y$. Here again x, y are parallel n-cells;

– we write $u \rhd_n v$ if $u : x \to_n y$ and $v : y \to_n z$ for some m-cells u, v and n-cells x, y, z;

– if $n > 0$ and u is an n-cell, we write $u^\flat$ for $\sigma_{0,n}(u)$ and $u^\sharp$ for $\tau_{0,n}(u)$, so that we get $u : u^\flat \to_0 u^\sharp$.

3.2 Strict ω-categories

A *strict ω-category* is a globular set C endowed with operations of composition and units, satisfying the laws of associativity, units and interchange, as follows:

– if u, v are m-cells such that $u \rhd_n v$, we write $u *_n v$ for the *n-composition* of u with v (in diagrammatic order);

– if x is an n-cell, we write $1_x : x \to x$ for the corresponding *n+1-dimensional unit*;

– if x is an n-cell and $m > n$, we write 1^m_x for the corresponding *m-dimensional unit*. We also write 1^n_x for x;

– if $m > n > p$, we write $u *_p v$ for $1^m_u *_p v$ whenever $u : x \to_p y$ is an n-cell and $v : y \to_p z$ is an m-cell;

– similarly, we write $u *_p v$ for $u *_p 1^m_v$ whenever $u : x \to_p y$ is an m-cell and $v : y \to_p z$ is an n-cell.

If $m > n$, the following identities hold for any m-cells $u \rhd_n v \rhd_n w$ and for any m-cell $u : x \to_n y$:

$$(u *_n v) *_n w = u *_n (v *_n w), \qquad 1^m_x *_n u = u = u *_n 1^m_y.$$

If $m > n > p$, the following identities hold for any m-cells $u \rhd_n u'$ and $v \rhd_n v'$ such that $u \rhd_p v$ (so that $u' \rhd_p v'$), for any n-cells $x \rhd_p y$, and for any p-cell z:

$$(u *_n u') *_p (v *_n v') = (u *_p v) *_n (u' *_p v'), \qquad 1^m_x *_p 1^m_y = 1^m_{x *_p y}, \qquad 1^m_{1^n_z} = 1^m_z.$$

An *ω-functor* is a morphism of globular sets preserving compositions and units. Thus, ω-categories and ω-functors build the category ω**Cat**, which is our main object of study.
The forgetful functor $U : \omega$**Cat** $\to$ **Glob** is finitary monadic [2] and **Glob** is a topos of presheaves on a small category: therefore ω**Cat** is complete and cocomplete. On the other hand, the left adjoint to U takes a globular set to the *free* ω-category it generates. In particular, consider $Y :$ **O** $\to$ **Glob** the Yoneda embedding: we get, for each n, a representable globular set $Y(n) = \mathbf{O}(-, n)$.

Definition 5. *For $n \geq 0$, the* n-globe $\mathbf{O}^n$ *is the free ω-category generated by $Y(n)$.*

Notice that $\mathbf{O}^n$ has exactly two non-identity i-cells for $i < n$, exactly one non-identity n-cell, and no non-identity cells in dimensions $i > n$.

Proposition 5. ω**Cat** *is locally finitely presentable.*

Proof. It is a general fact that the representable objects $Y(n)$ are *finitely presentable*. Because U preserves filtered colimits, all n-globes are finitely presentable objects in ω**Cat**. $\lhd$

3.3 Shift construction

The following construction will prove essential in defining the functor Γ of Section 4.4 below. Thus, given an ω-category C and two 0-cells x, y in it, we define a new ω-category $[x, y]$ as follows:

– there is an n-cell $[u]$ in $[x, y]$ for each n+1-cell $u : x \to_0 y$;

– for any n+1-cells $u, v : x \to_0 y$ and for any n+2-cell $w : u \to v$, we have $[w] : [u] \to [v]$ in $[x, y]$;

– n-composition is defined by $[u] *_n [v] = [u *_{n+1} v]$ whenever $u \rhd_{n+1} v$;

– m-dimensional units are defined by $1^m_{[u]} = \left[1^{m+1}_u\right]$.

The verification of the axioms of ω-categories is straightforward. We shall use some additional operations described below. For any 0-cells x, y, z, we get:

- a *precomposition* ω-functor $u \cdot - : [y, z] \to [x, z]$ for each 1-cell $u : x \to y$, defined by $u \cdot [v] = [u *_0 v]$;
- a *postcomposition* ω-functor $- \cdot v : [x, y] \to [x, z]$ for each 1-cell $v : y \to z$, defined by $[u] \cdot v = [u *_0 v]$;
- a *composition* ω*-bifunctor* $- \circledast - : [x, y] \times [y, z] \to [x, z]$, defined by $[u] \circledast [v] = [u *_0 v]$.

4 The folk model structure

The first step is to consider, for each n, the globular set $\partial Y(n)$ having the same cells as $Y(n)$ except for removing the unique n-cell. Thus $\partial Y(n)$ generates an ω-category $\partial \mathbf{O}^n$, the *boundary* of the n-globe, and we get an inclusion ω-functor

$$\mathbf{i}_n : \partial \mathbf{O}^n \to \mathbf{O}^n.$$

Notice that, for each n, we get a pushout:

$$\begin{tikzcd} \partial \mathbf{O}^n \arrow[r, "\mathbf{i}_n"] \arrow[d, "\mathbf{i}_n"'] & \mathbf{O}^n \arrow[d] \\ \mathbf{O}^n \arrow[r] & \partial \mathbf{O}^{n+1} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \tag{3}$$

The rest of this section is devoted to the construction of a combinatorial model structure on ω**Cat** where

$$I \stackrel{\text{def.}}{=} \{\mathbf{i}_n \mid n \in \mathbb{N}\}$$

is a set of generating cofibrations.

4.1 I-injectives

Notice that an ω-functor $f : X \to Y$ in $I{-}\mathrm{inj}$ can equivalently be characterised as verifying the following conditions:

- for any 0-cell y in Y, there is a 0-cell x in X such that $f\,x = y$;
- for any n-cells $x \parallel x'$ in X and for any $v : f\,x \to f\,x'$ in Y, there is $u : x \to x'$ in X such that $f\,u = v$.

Lemma 3. *An ω-functor $f : X \to Y$ in $I{-}\mathrm{inj}$ satisfies the following properties:*

- *for any n-cell y in Y, there is an n-cell x in X such that $f\,x = y$;*
- *for any n-cells $y \parallel y'$ in Y, there are n-cells $x \parallel x'$ in X such that $f\,x = y$ and $f\,x' = y'$.*

4.2 Omega-equivalence

Our definition of weak equivalences is based on two notions: reversible cells and ω-equivalence between parallel cells. These notions are defined by mutual coinduction.

Definition 6. *For any n-cells $x \parallel y$ in some ω-category:*

- *we say that x and y are* ω-equivalent*, and we write $x \sim y$, if there is a* reversible $n{+}1$*-cell $u : x \xrightarrow{\sim} y$;*
- *we say that the $n{+}1$-cell $u : x \to y$ is* reversible*, and we write $u : x \xrightarrow{\sim} y$, if there is an $n{+}1$-cell $\overline{u} : y \to x$ such that $u *_n \overline{u} \sim 1_x$ and $\overline{u} *_n u \sim 1_y$.*

Such a $\overline{u}$ is called a weak inverse *of u.*

Notice that there is no base case in such a definition. Hence, we get infinite trees of cells of increasing dimension. We now establish the first properties of reversible cells and ω-equivalence.

Lemma 4. *For any ω-functor $f : X \to Y$ and for any $u : x \xrightarrow{\sim} x'$ in X, we have $f\,u : f\,x \xrightarrow{\sim} f\,x'$ in Y. Hence, f preserves $\sim$.*

Proof. Suppose that x, x' are n-cells with $u : x \xrightarrow{\sim} x'$. By definition, there is an $n{+}1$-cell $\overline{u} : x' \to x$ such that $u *_n \overline{u} \sim 1_x$ and $\overline{u} *_n u \sim 1_{x'}$, whence reversible $n{+}2$-cells $v : u *_n \overline{u} \xrightarrow{\sim} 1_x$ and $v' : \overline{u} *_n u \xrightarrow{\sim} 1_{x'}$. Now, by coinduction, $f\,v : f\,u *_n f\,\overline{u} \xrightarrow{\sim} 1_{f\,x}$ and $f\,v' : f\,\overline{u} *_n f\,u \xrightarrow{\sim} 1_{f\,x'}$. Therefore $f\,u : f\,x \xrightarrow{\sim} f\,x'$. $\triangleleft$

Proposition 6. *The relation $\sim$ is an ω-congruence. More precisely:*

i. For any n-cell x, we get $1_x : x \xrightarrow{\sim} x$. Hence, $\sim$ is reflexive.

ii. For any reversible $n{+}1$-cell $u : x \xrightarrow{\sim} y$, we get $\overline{u} : y \xrightarrow{\sim} x$. Hence, $\sim$ is symmetric.

iii. For any reversible $n{+}1$-cells $u : x \xrightarrow{\sim} y$ and $v : y \xrightarrow{\sim} z$, we get $u *_n v : x \xrightarrow{\sim} z$. Hence, $\sim$ is transitive.

iv. For any n-cells x, y, z, and for any $u : x \to y$, $s, t : y \to_n z$ and $v : s \xrightarrow{\sim} t$ we get $u *_n v : u *_n s \xrightarrow{\sim} u *_n t$. There is a similar property for postcomposition. Hence, $\sim$ is compatible with compositions.

Proof. For (i), the proof is by coinduction, whereas (ii) follows immediately from the definition. Let x, y, z, u, v, s and t as in (iv), and consider f, the precomposition ω-functor $u \cdot - : [y, z] \to [x, z]$ (Section 3.3). As $v : s \xrightarrow{\sim} t$, we easily get $[v] : [s] \xrightarrow{\sim} [t]$, so that Lemma 4 applies and $f\,[v] : [s] \xrightarrow{\sim} [t]$, whence $u *_n v : u *_n s \xrightarrow{\sim} u *_n t$. The same holds for postcomposition. As for (iii), suppose that $u : x \xrightarrow{\sim} y$ and $v : y \xrightarrow{\sim} z$. By definition, there are $n{+}1$-cells $\overline{u} : y \to u$ and $\overline{v} : z \to y$ together with reversible $n{+}2$-cells $w : u *_n \overline{u} \xrightarrow{\sim} 1_x$ and $t : v *_n \overline{v} \xrightarrow{\sim} 1_y$. By using the compatibility property (iv) just established, we get $u *_n v *_n \overline{v} *_n \overline{u} \sim u *_n 1_y *_n \overline{u} = u *_n \overline{u}$. Also $u *_n \overline{u} \sim 1_x$. By coinduction, transitivity holds in dimension $n{+}1$, whence $u *_n v *_n \overline{v} *_n \overline{u} \sim 1_x$. Likewise $\overline{v} *_n \overline{u} *_n u *_n v \sim 1_z$. Therefore $u *_n v : x \xrightarrow{\sim} z$. $\triangleleft$

There is a convenient notion of weak uniqueness, related to ω-equivalence.

Definition 7. *A condition $\mathcal{C}$ defines a* weakly unique *cell $u : x \to y$ if we have $u \sim u'$ for any other $u' : x \to y$ satisfying $\mathcal{C}$.*

A less immediate, but crucial result is the following “weak division” property.

Lemma 5. *Any reversible 1-cell $u : x \xrightarrow{\sim} y$ satisfies the* left division property*:*

– *For any 1-cell $w : x \to z$, there is a weakly unique 1-cell $v : y \to z$ such that $u *_0 v \sim w$.*

– *For any 1-cells $s, t : y \to z$ and for any 2-cell $w : u *_0 s \to u *_0 t$, there is a weakly unique 2-cell $v : s \to t$ such that $u *_0 v \sim w$.*

– *More generally, for all $n > 0$, for any parallel n-cells $s, t : y \to_0 z$ and for any $n{+}1$-cell $w : u *_0 s \to u *_0 t$, there is a weakly unique $n{+}1$-cell $v : s \to t$ such that $u *_0 v \sim w$.*

Similarly, $u : x \xrightarrow{\sim} y$ satisfies the right division property*.*

In fact, this also applies to any reversible 2-cell $u : x \xrightarrow{\sim} y$, seen as a reversible 1-cell in the ω-category $[u^\flat, u^\sharp]$.

Proof. We have a weak inverse $\overline{u} : y \xrightarrow{\sim} x$ and some reversible 2-cell $r : \overline{u} *_0 u \xrightarrow{\sim} 1_y$.

– In the first case, we have $u *_0 v \sim w$ if and only if $v \sim \overline{u} *_0 w$.

– In the second case, $u *_0 v \sim w$ implies $(r *_0 s) *_1 v = (\overline{u} *_0 u *_0 v) *_1 (r *_0 t) \sim (\overline{u} *_0 w) *_1 (r *_0 t)$ by interchange and compatibility. By left division by $r *_0 s$ (first case), this condition defines a weakly unique v. Hence, we get weak uniqueness for left division by u. Moreover, this condition implies $\overline{u} *_0 u *_0 v \sim \overline{u} *_0 w$ by right division by $r *_0 t$ (first case), from which we get $u *_0 v \sim w$ by weak uniqueness applied to $\overline{u}$.

– The general case (for left and right division) is proved in the same way by induction on n. $\triangleleft$

4.3 ω-Weak equivalences

If we replace equality by ω-equivalence in the definition of I-injectives, we get ω-weak equivalences.

Definition 8. *An ω-functor $f : X \to Y$ is an ω-weak equivalence whenever it satisfies the following conditions:*

i. for any 0-cell y in Y, there is a 0-cell x in X such that $f\,x \sim y$;

ii. for any n-cells $x \parallel x'$ in X and for any $v : f\,x \to f\,x'$ in Y, there is $u : x \to x'$ in X such that $f\,u \sim v$.

We write $\mathcal{W}$ for the class of ω-weak equivalences.

Remark 5. As equality implies ω-equivalence (Proposition 6), we have

$$I\text{−inj} \subseteq \mathcal{W},$$

which is exactly condition **(S2)** of Theorem 1. $\Diamond$

We first remark that ω-equivalences are *weakly injective*, in the sense of the following Lemma.

Lemma 6. *If $f : X \to Y$ is in $\mathcal{W}$, then $x \sim x'$ for any $x \parallel x'$ in X such that $f\,x \sim f\,x'$ in Y.*

Proof. Let $f \in \mathcal{W}$ and x, x' parallel n-cells such that $f\,x \sim f\,x'$. There are $n{+}1$-cells $u : f\,x \to f\,x'$ and $\overline{u} : f\,x' \to f\,x$ such that $u *_n \overline{u} \sim 1_{f\,x}$ and $\overline{u} *_n u \sim 1_{f\,x'}$. Because f is a ω-weak equivalence, we get $n{+}1$-cells $v : x \to x'$ and $\overline{v} : x' \to x$ such that $f\,v \sim u$ and $f\,\overline{v} \sim \overline{u}$. By using Proposition 6,(iii) and (iv), and the preservation of compositions and units by f,

$$\begin{aligned} f(v *_n \overline{v}) &\sim u *_n \overline{u} \\ &\sim f(1_x) \end{aligned}$$

By coinduction, $v *_n \overline{v} \sim 1_x$ and likewise $\overline{v} *_n v \sim 1_{x'}$, whence $x \sim x'$. $\triangleleft$

The “3 for 2” property states that whenever two ω-functors out of f, g and $h = g \circ f$ are ω-weak equivalences, then so is the third. So there are really three statements, that we shall address separately.

Lemma 7. *Let $f : X \to Y$ and $g : Y \to Z$ be ω-weak equivalences. Then $g \circ f : X \to Z$ is in $\mathcal{W}$.*

Proof. Suppose that $f : X \to Y$, $g : Y \to Z$ are ω-weak equivalences and let $h = g \circ f$. If z is a 0-cell in Z, there is a 0-cell y in Y such that $g\,y \sim z$, and a 0-cell x in X such that $f\,x \sim y$. By Lemma 4, $h\,x \sim g\,y$, and by Proposition 6,(iii), $h\,x \sim z$. Now, let x, x' be two parallel n-cells in X and $w : h\,x \to h\,x'$ be an $n{+}1$-cell in Z. There is a $v : f\,x \to f\,x'$ such that $g\,v \sim w$ and a $u : x \to x'$ such that $f\,u \sim v$. By Lemma 4 and Proposition 6,(iii) again, we get $h\,u \sim w$ and we are done. $\triangleleft$

Lemma 8. *Let $f : X \to Y$, $g : Y \to Z$ be ω-functors and suppose that g and $g \circ f$ are ω-weak equivalences. Then f is in $\mathcal{W}$.*

Proof. Let f, g and $h = g \circ f$ such that $g \in \mathcal{W}$ and $h \in \mathcal{W}$. Let y be a 0-cell in Y, and $z = g\,y$. There is a 0-cell x in X such that $h\,x \sim z$. By Lemma 6, $f\,x \sim y$. Likewise, let x, x' be parallel n-cells in X and $v : f\,x \to f\,x'$ an $n{+}1$-cell in Y. We get $g\,v : h\,x \to h\,x'$, therefore there is an $n{+}1$-cell $u : x \to x'$ in X such that $h\,u \sim g\,v$. By Lemma 6, $f\,u \sim v$ and we are done. $\triangleleft$

The remaining part of the 3-for-2 property for $\mathcal{W}$ is significantly harder to show and will be addressed in Section 4.5.

Lemma 9. *The class $\mathcal{W}$ is closed under retract and transfinite composition.*

Proof. The closure under retracts follows immediately from the definition, by using Lemma 4.
As for the closure under transfinite composition, let $\alpha > 0$ be an ordinal, viewed as a category with a unique morphism $\beta \to \gamma$ for each pair $\beta \le \gamma$ of ordinals $< \alpha$, and $X : \alpha \to \omega\mathbf{Cat}$ a functor, preserving colimits. We denote by w_β^γ the morphism $X(\beta \to \gamma) : X(\beta) \to X(\gamma)$, and by $(\overline{X}, w_\beta)$ the colimit of the directed system $(X(\beta), w_\beta^\gamma)$. Suppose that each $w_\beta^{\beta+1}$ belongs to $\mathcal{W}$. We need to show that $w_0 : X(0) \to \overline{X}$ is still a ω-weak equivalence. We first establish that for each $\beta < \alpha$, $w_0^\beta \in \mathcal{W}$, by induction on β:

- if $\beta = 0$, w_0^β is the identity on $X(0)$, thus belongs to $\mathcal{W}$;
- if β is a successor ordinal, $\beta = \gamma + 1$ and $w_0^\beta = w_\gamma^{\gamma+1} \circ w_0^\gamma$. By induction, $w_0^\gamma \in \mathcal{W}$, and by hypothesis $w_\gamma^{\gamma+1} \in \mathcal{W}$, hence the result, by composition;
- if β is a limit ordinal, $\beta = \sup_{\gamma<\beta} \gamma$. Let $n > 0$, (x, y) a pair of parallel $n{-}1$ cells in $X(0)$ and $u : w_0^\beta(x) \to w_0^\beta(y)$ an n-cell in $X(\beta)$. Because X preserves colimits, there is already a $\gamma < \beta$ and an n-cell v in $X(\gamma)$ such that $v : w_0^\gamma(x) \to w_0^\gamma(y)$ and $w_\gamma^\beta(v) = u$. By the induction hypothesis, w_0^γ is a ω-weak equivalence, and there is a $z : x \to y$ in $X(0)$ such that $w_0^\gamma(z) \sim v$. By composing with w_γ^β, we get $w_0^\beta(z) \sim u$. The same argument applies to the case $n = 0$, so that $w_0^\beta \in \mathcal{W}$.

Now we complete the proof by induction on α itself: if α is a successor ordinal, then $\alpha = \beta + 1$ and w_0 is w_0^β, hence belongs to $\mathcal{W}$, as we just proved. If α is a limit ordinal, we reproduce the argument of the limit case above, using again the fact that w_0^β is a ω-weak equivalence for any $\beta < \alpha$. ◁

Corollary 1. *$I{-}\mathrm{cof} \cap \mathcal{W}$ is closed under retract and transfinite composition.*

4.4 Cylinders

The proofs of condition **(S2)** , part of **(S1)** and **(S3)** were directly based on our definitions of generating cofibrations and ω-weak equivalences. As for the remaining points, we shall need a new construction: to each ω-category X we associate an ω-category $\Gamma(X)$ whose cells are the *reversible cylinders* of X. The correspondence Γ turns out to be functorial and endowed with natural transformations from and to the identity functor. Reversible cylinders are in fact cylinders in the sense of [19] and [18], satifying an additional reversibility condition. In the present work, “cylinder” means “reversible cylinder”, as the general case will not occur.

Definition 9. *By induction on n, we define the notion of* n-cylinder $U : x \curvearrowright y$ *between n-cells x and y in some ω-category:*

- *a 0-cylinder $U : x \curvearrowright y$ in X is given by a reversible 1-cell $U^\natural : x \xrightarrow{\sim} y$;*
- *if $n > 0$, an n-cylinder $U : x \curvearrowright y$ in X is given by two reversible 1-cells $U^\flat : x^\flat \xrightarrow{\sim} y^\flat$ and $U^\sharp : x^\sharp \xrightarrow{\sim} y^\sharp$, together with some $n{-}1$-cylinder $[U] : [x] \cdot U^\sharp \curvearrowright U^\flat \cdot [y]$ in the ω-category $[x^\flat, y^\sharp]$.*

If $U : x \curvearrowright y$ is an n-cylinder, we write $\pi^1\, U$ and $\pi^2\, U$ for the n-cells x and y.

$$\begin{tikzcd} x \arrow[d, "U^\natural"'] \\ y \end{tikzcd} \qquad\qquad \begin{tikzcd} x^\flat \arrow[r, "x"] \arrow[d, "U^\flat"'] & x^\sharp \arrow[d, "U^\sharp"] \arrow[dl, "U^\natural"'] \\ y^\flat \arrow[r, "y"'] & y^\sharp \end{tikzcd}$$

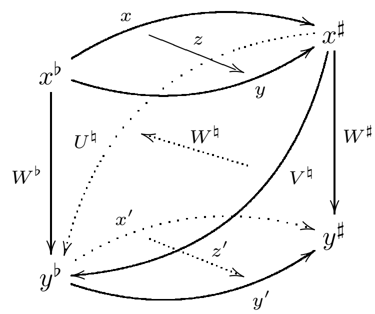
We also write $\pi_X^1\, U$ and $\pi_X^2\, U$ to emphasize the fact that U is an n-cylinder in the ω-category X. The next step is to show that n-cylinders in X are the n-cells of a globular set.

Definition 10. *By induction on n, we define the* source n-cylinder $U : x \curvearrowright x'$ *and the* target n-cylinder $V : y \curvearrowright y'$ *of any $n{+}1$-cylinder $W : z \curvearrowright z'$ between $n{+}1$-cells $z : x \to y$ and $z' : x' \to y'$:*

– *if $n = 0$, then $U^\natural = W^\flat$ and $V^\natural = W^\sharp$;*

– *if $n > 0$, then $U^\flat = V^\flat = W^\flat$ and $U^\sharp = V^\sharp = W^\sharp$, whereas the two $n{-}1$-cylinders $[U]$ and $[V]$ are respectively defined as the source and the target of the n-cylinder $[W]$ in the ω-category $[z^\flat, z'^\sharp]$.*

In that case, we write $W : U \to V$ or also $W : U \to V \mid z \curvearrowright z'$.



Lemma 10. *We have $U \parallel V$ for any $n{+}1$-cylinder $W : U \to V$. In other words, cylinders form a globular set.*

Proof. By induction on n. ◁

Remark that the 0-source U and the 0-target V of an $n{+}1$-cylinder W are given by $U^\natural = W^\flat$ and $V^\natural = W^\sharp$.

We now define *trivial cylinders*.

Definition 11. *By induction on n, we define the* trivial n-cylinder *$\tau\, x : x \curvearrowright x$ for any n-cell x:*

– *if $n = 0$, then $(\tau\, x)^\natural = 1_x$;*

– *if $n > 0$, then $(\tau\, x)^\flat = 1_{x^\flat}$ and $(\tau\, x)^\sharp = 1_{x^\sharp}$, whereas $[\tau\, x]$ is the trivial cylinder $\tau[x]$ in $[x^\flat, x^\sharp]$.*

We also write $\tau_X\, x$ for $\tau\, x$ to emphasize the fact that x is an n-cell of the ω-category X. The following result is a straightforward consequence of the definition.

Lemma 11. *We have $\tau\, x \parallel \tau\, y$ for any n-cells $x \parallel y$, and $\tau\, z : \tau\, x \to \tau\, y$ for any $z : x \to y$.*

More generally, we get the following notion of *degenerate cylinder*:

Definition 12. *An n-cylinder between parallel cells is* degenerate *whenever $n = 0$ or $n > 0$ and its source and target are trivial.*

Remark that $\tau\, x \parallel U \parallel \tau\, y$ for any degenerate n-cylinder $U : x \curvearrowright y$. The next easy lemma gives a more concrete description of degenerate cylinders:

Lemma 12. *i. For any degenerate n-cylinder $U : x \curvearrowright y$, we get a reversible $n{+}1$-cell $U^\natural : x \xrightarrow{\sim} y$.*

ii. Conversely, any reversible $n{+}1$-cell $u : x \xrightarrow{\sim} y$ corresponds to a unique degenerate n-cylinder $U : x \curvearrowright y$.

In particular, the trivial n-cylinder $\tau\, x : x \curvearrowright x$ is the degenerate n-cylinder given by $(\tau\, x)^\natural = 1_x : x \xrightarrow{\sim} x$.

Thus, for each ω-category X, we have defined a globular set $\Gamma(X)$ whose n-cells are n-cylinders in X, together with globular morphisms $\pi^1_X, \pi^2_X : \Gamma(X) \to X$ and $\tau_X : X \to \Gamma(X)$ such that $\pi^1_X \circ \tau_X = \mathrm{id}_X = \pi^2_X \circ \tau_X$.

$$\begin{tikzcd}
 & X \arrow[dl, "\mathrm{id}_X"'] \arrow[d, "\tau_X"] \arrow[dr, "\mathrm{id}_X"] & \\
X & \Gamma(X) \arrow[l, "\pi^1_X"] \arrow[r, "\pi^2_X"'] & X
\end{tikzcd}$$

Now we may define compositions of n-cylinders in X, as well as units, in such a way that the globular set $\Gamma(X)$ becomes an ω-category: this is done in detail in appendix A (see also [19] and [18]). Thus, from now on, $\Gamma(X)$ denotes this ω-category. Likewise, π^1_X, π^2_X and τ_X become ω-functors. The following theorem, proved in appendix, summarizes the properties we actually use in the construction of our model structure.

Theorem 2. *The correspondence $X \mapsto \Gamma(X)$ is the object part of an endofunctor on ω***Cat***, and $\pi^1, \pi^2 : \Gamma \to \mathrm{id}$, $\tau : \mathrm{id} \to \Gamma$ are natural transformations.*

In particular, we get $f\,U : f\,x \curvearrowright f\,x'$ for any ω-functor $f : X \to Y$ and for any n-cylinder $U : x \curvearrowright x'$ in X.

We end this presentation of n-cylinders with the following important "transport" lemma.

Lemma 13. *For any parallel n-cylinders $U : x \curvearrowright x'$ and $V : y \curvearrowright y'$, we have a* topdown transport*:*

i. For any $z : x \to y$, there is $z' : x' \to y'$ together with a cylinder $W : U \to V \mid z \curvearrowright z'$.

ii. Such a z' is weakly unique: $z' \sim z''$ for any $z'' : x' \to y'$ together with a cylinder $W' : U \to V \mid z \curvearrowright z''$.

iii. Conversely, there is a cylinder $W' : U \to V \mid z \curvearrowright z''$ for any $z'' : x' \to y'$ such that $z' \sim z''$.

Similarly, we have a bottom up transport.

Proof. We proceed by induction on n.

- If $n = 0$, let $U : x \curvearrowright x'$ and $V : y \curvearrowright y'$ be parallel 0-cylinders, and a 1-cell $z : x \to y$. By definition, there are reversible 1-cells $u : x \to x'$ and $v : y \to y'$. Let $\overline{u} : x' \to x$ a weak inverse of u, and define $z' = \overline{u} *_0 z *_0 v$. Now $u *_0 z' = u *_0 \overline{u} *_0 z *_0 v$. As $u *_0 \overline{u} \sim 1_x$, $u *_0 z' \sim z *_0 v$, by using Proposition (6). Whence a reversible 2-cell $w : z *_0 v \xrightarrow{\sim} u *_0 z'$, that is a reversible 1-cell, or 0-cylinder, in the ω-category $[x, y']$. Thus we get a 1-cylinder $W : U \to V \mid z \curvearrowright z'$, and (i) is proved. Suppose now that there is a $z'' : x' \to y'$ together with a 1-cylinder $W' : U \to V \mid z \curvearrowright z''$. It follows that $u *_0 z' \sim z *_0 v \sim u *_0 z''$, whence $z' \sim z''$ by Lemma 5. This proves (ii). Suppose finally that $z'' \sim z'$. We get $u *_0 z'' \sim u *_0 z' \sim z *_0 v$, and a cylinder $W' : U \to V \mid z \curvearrowright z''$ as above, which proves (iii).
- Suppose that (i), (ii) and (iii) hold in dimension n. Let $U : x \curvearrowright x'$, $V : y \curvearrowright y'$ parallel $n{+}1$-cylinders and $z : x \to y$ an $n{+}2$-cell. By definition, we have reversible 1-cells $U^\flat = V^\flat : x^\flat \xrightarrow{\sim} x'^\flat$, $U^\sharp = V^\sharp : x^\sharp \xrightarrow{\sim} x'^\sharp$, together with parallel n-cylinders $[U] : [x] \cdot U^\sharp \curvearrowright U^\flat \cdot [x']$ and $[V] : [y] \cdot y^\sharp \curvearrowright V^\flat \cdot [y']$ in $[x^\flat, y'^\sharp]$. Now $[z] \cdot U^\sharp : [x] \cdot U^\sharp \to [y] \cdot V^\sharp$ is an $n{+}1$-cell $[w]$ in $[x^\flat, y'^\sharp]$. By the induction hypothesis, we get an $n{+}1$-cell $[w'] : U^\flat \cdot [x'] \to V^\flat \cdot [y']$ and an $n{+}1$-cylinder $[W_0] : [U] \to [V] \mid [w] \curvearrowright [w']$ in $[x^\flat, y'^\sharp]$. By Lemma 5, there is a $[z'] : [x'] \to [y']$ such that $[w'] \sim U^\flat \cdot [z']$. Thus, part (iii) of the induction hypothesis gives an $n{+}1$-cylinder $[W] : [U] \to [V] \mid [z] \cdot U^\sharp \curvearrowright U^\flat \cdot [z']$. But this defines an $n{+}2$-cylinder $W : U \to V \mid z \curvearrowright z'$, and (i) holds in dimension $n{+}1$. Moreover, by induction, the above cell $[w']$ is weakly unique, and so is z', by Lemma 5: this gives (ii) in dimension $n{+}1$. Finally, if $z'' \sim z'$, $U^\flat \cdot [z''] \sim U^\flat \cdot [z']$ in $[x^\flat, y'^\sharp]$, and the induction hypothesis gives an $n{+}1$-cylinder $[W'] : [U] \to [V] \mid [z] \cdot U^\sharp \curvearrowright U^\flat \cdot [z'']$, whence an $n{+}2$-cylinder $W' : U \to V \mid z \curvearrowright z''$, so that (iii) holds in dimension $n{+}1$. $\triangleleft$

Corollary 2. *For each ω-category X, π^1_X, π^2_X are in $I{-}\mathrm{inj}$ and τ_X is in $\mathcal{W}$.*

Proof. Let $U \mid x \curvearrowright x'$ and $V \mid y \curvearrowright y'$ be parallel n-cylinders in X and $z : \pi^1_X U \to \pi^1_X V$ an $n{+}1$-cell. By Lemma 13, there is an $n{+}1$-cylinder $W : U \to V$ such that $\pi^1_X W = z$. This proves that π^1_X is in $I{-}\mathrm{inj}$. Likewise, by bottom up transport, π^2_X is in $I{-}\mathrm{inj}$. But $I{-}\mathrm{inj} \subseteq \mathcal{W}$ by **(S2)** so that π^1_X is a ω-weak equivalence. Now $\pi^1_X \circ \tau_X = \mathrm{id}_X$, and by Lemma 8, $\tau_X \in \mathcal{W}$. $\triangleleft$

4.5 Gluing factorization

For any ω-functor $f : X \to Y$, we consider the following pullback:

$$\begin{tikzcd}
\Pi(f) \arrow[r, "f^*\pi^1_Y"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
\Gamma(Y) \arrow[r, "\pi^1_Y"'] & Y
\end{tikzcd}$$

We write $\hat{f} : \Pi(f) \to Y$ for $\pi^2 \circ f'$, so that the following diagram commutes:

$$
\begin{tikzcd}
X \arrow[r, "\tilde f"] \arrow[d, "f"'] \arrow[rr, bend left, "\mathrm{id}_X"] & \Pi(f) \arrow[r, "f^*\pi^1_Y"] \arrow[d, "f'"] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
Y \arrow[r, "\tau_Y"'] \arrow[rr, bend right, "\mathrm{id}_Y"'] & \Gamma(Y) \arrow[r, "\pi^1_Y"'] & Y
\end{tikzcd}
$$

Since π^1_Y is in $I-\mathrm{inj}$, so is its pullback $f^*\pi^1_Y$. By **(S2)** , $f^*\pi^1_Y$ is in $\mathcal{W}$. As $f^*\pi^1_Y \circ \tilde f = \mathrm{id}_X$, by Lemma 8, $\tilde f$ is also a ω-weak equivalence.

Definition 13. *The decomposition $f = \hat f \circ \tilde f$ is called the* gluing factorization *of f.*

$$
\begin{tikzcd}
X \arrow[r, "\tilde f"] \arrow[rr, bend right, "f"'] & \Pi(f) \arrow[r, "\hat f"] & Y
\end{tikzcd}
$$

The above constructions may be described more concretely as follows:

- an n-cell in $\Pi(f)$ is a pair (x, U) where x is an n-cell in X and $U : f\,x \curvearrowright y$ is an n-cylinder in Y;
- $\tilde f\,x = (x, \tau\, f\, x)$ for any n-cell x in X, and $\hat f(x, U) = \pi^2\, U = y$ for any n-cylinder $U : f\,x \curvearrowright y$ in Y.

The gluing factorization leads to an extremely useful characterization of ω-weak equivalences.

Proposition 7. *An ω-functor $f : X \to Y$ is in $\mathcal{W}$ if and only if $\hat f : \Pi(f) \to Y$ is in $I-\mathrm{inj}$.*

Proof. Suppose that $\hat f$ is in $I-\mathrm{inj}$, then it is in $\mathcal{W}$ by **(S2)** ; as $\tilde f$ is a ω-weak equivalence, so is the composition $f = \hat f \circ \tilde f$, by Lemma 7. Conversely, suppose that f is in $\mathcal{W}$, and let us show that $\hat f$ is in $I-\mathrm{inj}$:

- For any 0-cell y in Y, there is a 0-cell x in X such that $f\,x \sim y$. Hence, we get a reversible 1-cell $u : f\,x \xrightarrow{\sim} y$ defining a 0-cylinder $U : f\,x \curvearrowright y$, so that (x, U) is a 0-cell in $\Pi(f)$ and $\hat f(x, U) = y$.
- For any n-cells $(x, T) \parallel (x', T')$ in $\Pi(f)$, we get parallel n-cylinders $T : f\,x \curvearrowright y$ and $T' : f\,x' \curvearrowright y'$. For any $n{+}1$-cell $w : y \to y'$, Lemma 13, bottom up direction, gives $v : f\,x \to f\,x'$ together with $V : T \to T' \mid v \curvearrowright w$. Since f is in $\mathcal{W}$ and $x \parallel x'$, we get an $n{+}1$-cell $u : x \to x'$ such that $f\,u \sim v$. By Lemma 13, (iii), bottom up direction, we get $U : T \to T' \mid f\,u \curvearrowright w$, so that $(u, U) : (x, T) \to (x', T')$ is an $n{+}1$-cell in $\Pi(f)$ and $\hat f(u, U) = w$. ◁

Corollary 3. *$\mathcal{W}$ is the smallest class containing $I-\mathrm{inj}$ which is closed under composition and right inverse.*

It is now possible to prove the remaining part of condition 3-for-2 for $\mathcal{W}$.

Lemma 14. *If $f : X \to Y$ and $h = g \circ f : X \to Z$ are in $\mathcal{W}$, so is $g : Y \to Z$.*

Proof.
- For any 0-cell z in Z, there is a 0-cell x in X such that $h\,x \sim z$. So we get $g\,y \sim z$, where $y = f\,x$.
- Let $y \parallel y'$ be n-cells in Y, and let $w : g\,y \to g\,y'$ be an $n{+}1$-cell in Z.
  + By Proposition 7, $\hat f$ is in $I-\mathrm{inj}$, so that Lemma 3 applies, and we get $x \parallel x'$ in X and parallel n-cylinders $T : f\,x \curvearrowright y$ and $T' : f\,x' \curvearrowright y'$.
  + By Theorem 2, we get parallel n-cylinders $g\,T : h\,x \curvearrowright g\,y$ and $g\,T' : h\,x' \curvearrowright g\,y'$.
  + By Proposition 7, $\hat h$ is in $I-\mathrm{inj}$ and we get $u : x \to x'$ together with $U : g\,T \to g\,T' \mid h\,u \curvearrowright w$.
  + By Lemma 13, (i) we get $v : y \to y'$ together with $V : T \to T' \mid f\,u \curvearrowright v$.
  + By Theorem 2, we get $g\,V : g\,T \to g\,T' \mid h\,u \curvearrowright g\,v$.
  + By Lemma 13, (ii), we get $g\,v \sim w$. ◁

4.6 Immersions

In order to complete the proof of condition **(S3)** , we introduce a new class of ω-functors.

Definition 14. *An* immersion *is an ω-functor $f : X \to Y$ satisfying the following three conditions:*

(Z1) *there is a retraction $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$;*

(Z2) *there is an ω-functor $h : Y \to \Gamma(Y)$ such that $\pi^1_Y \circ h = f \circ g$ and $\pi^2_Y \circ h = \mathrm{id}_Y$;*

(Z3) *$h \circ f = \tau_Y \circ f$. In other words, h is trivial on $f(X)$.*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "\mathrm{id}_X"'] & Y \arrow[r, "g"] & X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[d, "f"'] & Y \arrow[l, "g"'] \arrow[d, "h"] \arrow[dr, bend left, "\mathrm{id}_Y"] & \\ Y & \Gamma(Y) \arrow[l, "\pi^1_Y"] \arrow[r, "\pi^2_Y"'] & Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "f"'] & Y \arrow[d, "h"] \\ Y \arrow[r, "\tau_Y"'] & \Gamma(Y) \end{tikzcd}$$

We write $\mathcal{Z}$ for the class of immersions.

Notice that, by naturality of τ, condition **(Z3)** can be replaced by the following one:

(Z3') $h \circ f = \Gamma(f) \circ \tau_X$.

The gluing construction of the previous section yields a characterization of immersions by a lifting property.

Lemma 15. *An ω-functor $f : X \to Y$ is an immersion if and only if there is an ω-functor $k : Y \to \Pi(f)$ such that $k \circ f = \tilde{f}$ and $\hat{f} \circ k = \mathrm{id}_Y$.*

$$\begin{tikzcd} X \arrow[r, "\tilde{f}"] \arrow[d, "f"'] & \Pi(f) \arrow[d, "\hat{f}"] \\ Y \arrow[r, "\mathrm{id}_Y"'] \arrow[ur, dashed, "k"] & Y \end{tikzcd}$$

Proof. Let $f : X \to Y$, and suppose that there is a $k : Y \to \Pi(f)$ satisfying the above lifting property. Define $g = f^*\pi^1_Y \circ k$ and $h = f' \circ k$. We get $g \circ f = f^*\pi^1_Y \circ k \circ f = f^*\pi^1_Y \circ \tilde{f} = \mathrm{id}_X$, hence **(Z1)** . Also $\pi^1_Y \circ h = \pi^1_Y \circ f' \circ k = f \circ f^*\pi^1_Y \circ k = f \circ g$ and $\pi^2_Y \circ h = \pi^2_Y \circ f' \circ k = \hat{f} \circ k = \mathrm{id}_Y$, hence **(Z2)** . Finally $h \circ f = f' \circ k \circ f = f' \circ \tilde{f} = \tau_Y \circ f$, hence **(Z3)** .
Conversely, suppose that $f : X \to Y$ is an immersion, and let g, h satisfy the conditions of Definition 14. By **(Z2)** , $\pi^1_Y \circ h = f \circ g$, so that the universal property of $\Pi(f)$ yields a unique $k : Y \to \Pi(f)$ such that $f^*\pi^1_Y \circ k = g$ and $f' \circ k = h$. Thus $\hat{f} \circ k = \pi^2_Y \circ f' \circ k = \pi^2_Y \circ h = \mathrm{id}_Y$, by **(Z2)** . Now $f^*\pi^1_Y \circ k \circ f = g \circ f = \mathrm{id}_X = f^*\pi^1_Y \circ \tilde{f}$ and $f' \circ k \circ f = h \circ f = \tau_Y \circ f$ by **(Z3)** so that $f' \circ k \circ f = f' \circ \tilde{f}$: by the universal property of $\Pi(f)$, this gives $k \circ f = \tilde{f}$, and we are done. ◁

Corollary 4. $I\text{−cof} \cap \mathcal{W} \subseteq \mathcal{Z}$.

Proof. Suppose that $f : X \to Y$ belongs to $I\text{−cof} \cap \mathcal{W}$. As $f \in \mathcal{W}$, by Proposition 7, $\hat{f} \in I\text{−inj}$. Now $f \in Cof$ has the left lifting property with respect to $\hat{f}$, so that there is a k such that $k \circ f = \tilde{f}$ and $\hat{f} \circ k = \mathrm{id}_Y$. By Lemma 15, f is an immersion. ◁

Lemma 16. $\mathcal{Z} \subset \mathcal{W}$.

Proof. Suppose that $f : X \to Y$ is an immersion, and let g, h as in Definition 14:

- For any 0-cell y in Y, we get $h\,y : f\,x \curvearrowright y$ where $x = g\,y$. Hence, we get $(h\,y)^\natural : f\,x \xrightarrow{\sim} y$, so that $f\,x \sim y$.
- For any n-cells $x \parallel x'$ in X and for any $v : f\,x \to f\,x'$ in Y, we have $h\,v : f\,u \curvearrowright v$ where $u = g\,v : x \curvearrowright x'$. By **(Z3)** , the cylinder $h\,v : \tau\,f\,x \to \tau\,f\,x'$ is degenerate. Hence, we get $(h\,v)^\natural : f\,u \xrightarrow{\sim} v$, so that $f\,u \sim v$.◁

Lemma 17. *$\mathcal{Z}$ is closed under pushout.*

Proof. Let $f : X \to Y$ be an immersion, $i : X \to X'$ an ω-functor and $f' : X' \to Y'$ the pushout of f by i:

$$\begin{tikzcd}
X \arrow[r, "i"] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\
Y \arrow[r, "j"'] \arrow[ur, phantom, "\ulcorner" very near end] & Y'
\end{tikzcd}$$

Since f is an immersion, we have $g : Y \to X$ and $h : Y \to \Gamma(Y)$ satisfying conditions **(Z1)** to **(Z3)** . By universality of the pushout and by **(Z3')** , we get $g' : Y' \to X'$ and $h' : Y' \to \Gamma(Y')$ such that the following diagrams commute:

$$\begin{tikzcd}
X \arrow[r, "i"] \arrow[d, "f"'] \arrow[dd, bend right=60, "\mathrm{id}_X"'] & X' \arrow[d, "f'"] \arrow[dd, bend left=60, "\mathrm{id}_{X'}"] \\
Y \arrow[r, "j"'] \arrow[d, "g"'] \arrow[ur, phantom, "\ulcorner" very near end] & Y' \arrow[d, dashed, "g'"] \\
X \arrow[r, "i"'] & X'
\end{tikzcd}
\qquad
\begin{tikzcd}
 & X \arrow[r, "i"] \arrow[d, "f"'] \arrow[dl, bend right, "\tau_X"'] & X' \arrow[d, "f'"] \arrow[dr, bend left, "\tau_{X'}"] & \\
\Gamma(X) \arrow[dr, bend right, "\Gamma(f)"'] & Y \arrow[r, "j"'] \arrow[d, "h"'] \arrow[ur, phantom, "\ulcorner" very near end] & Y' \arrow[d, dashed, "h'"] & \Gamma(X') \arrow[dl, bend left, "\Gamma(f')"] \\
 & \Gamma(Y) \arrow[r, "\Gamma(j)"'] & \Gamma(Y') &
\end{tikzcd}$$

Finally, conditions **(Z1)** to **(Z3)** for g' and h' follow from conditions **(Z1)** to **(Z3)** for g and h. $\triangleleft$

Corollary 5. *$I\!-\!\mathrm{cof} \cap \mathcal{W}$ is closed under pushout.*

Proof. Let $f \in I\!-\!\mathrm{cof} \cap \mathcal{W}$ and f' a pushout of f. By Corollary 4, f is an immersion, and so is f' by Lemma 17. By Lemma 16, f' is a ω-weak equivalence. Now $I\!-\!\mathrm{cof}$ is stable by pushout, so that $f' \in I\!-\!\mathrm{cof}$. Hence $f' \in I\!-\!\mathrm{cof} \cap \mathcal{W}$ and we are done. $\triangleleft$

4.7 Generic squares

By Yoneda's Lemma, for each n, the functor $X \mapsto X_n$, from ω**Cat** to **Sets** is represented by the n-globe $\mathbf{O}^n$. Thus, to each n-cell x of X corresponds a unique ω-functor

$$\langle x \rangle : \mathbf{O}^n \to X.$$

Moreover, for any pair x, x' of n-cells in X, the condition of parallelism $x \parallel x'$ is equivalent to $\langle x \rangle \circ \mathbf{i}_n = \langle x' \rangle \circ \mathbf{i}_n$. By the pushout square (3) mentioned at the beginning of Section 4, we get a unique ω-functor

$$\langle x, x' \rangle : \partial\mathbf{O}^{n+1} \to X.$$

associated to any pair x, x' of parallel n-cells. This applies in particular to the case where $x = x' = o$, the unique proper n-cell of $\mathbf{O}^n$. The corresponding ω-functor is denoted by $\mathbf{o}_n = \langle o, o \rangle : \partial\mathbf{O}^{n+1} \to \mathbf{O}^n$. Since ω**Cat** is locally presentable, there is a factorization $\mathbf{o}_n = \mathbf{p}_n \circ \mathbf{k}_n$ with $\mathbf{p}_n \in I\!-\!\mathrm{inj}$ and $\mathbf{k}_n \in I\!-\!\mathrm{cof}$.

$$\begin{tikzcd}
\partial\mathbf{O}^{n+1} \arrow[r, "\mathbf{k}_n"] \arrow[rr, bend right, "\mathbf{o}_n"'] & \mathbf{P}^n \arrow[r, "\mathbf{p}_n"] & \mathbf{O}^n
\end{tikzcd}$$

Now by composition of $\mathbf{k}_n$ with both ω-functors $\mathbf{O}^n \to \partial\mathbf{O}^{n+1}$ of the pushout (3), we get $\mathbf{j}_n, \mathbf{j}'_n : \mathbf{O}^n \to \mathbf{P}^n$ such that the following diagram commutes:

$$\begin{tikzcd}
\partial\mathbf{O}^n \arrow[r, "\mathbf{i}_n"] \arrow[d, "\mathbf{i}_n"'] & \mathbf{O}^n \arrow[d] \arrow[dr, bend left, "\mathbf{j}_n"] \arrow[drr, bend left, "\mathrm{id}_{\mathbf{O}^n}"] & & \\
\mathbf{O}^n \arrow[r] \arrow[rr, bend right, "\mathbf{j}'_n"'] \arrow[rrr, bend right=40, "\mathrm{id}_{\mathbf{O}^n}"'] \arrow[ur, phantom, "\ulcorner" very near end] & \partial\mathbf{O}^{n+1} \arrow[r, "\mathbf{k}_n"] & \mathbf{P}^n \arrow[r, "\mathbf{p}_n"] & \mathbf{O}^n
\end{tikzcd}$$

The following definition singles out an important part of the above diagram.

Definition 15. *The* generic n-square *is the following commutative square:*

$$\begin{tikzcd}
\partial\mathbf{O}^n \arrow[r, "\mathbf{i}_n"] \arrow[d, "\mathbf{i}_n"'] & \mathbf{O}^n \arrow[d, "\mathbf{j}_n"] \\
\mathbf{O}^n \arrow[r, "\mathbf{j}'_n"'] & \mathbf{P}^n
\end{tikzcd}$$

Remark 6. Notice that $\mathbf{p}_n$ is in $I-\mathrm{inj}$, hence in $\mathcal{W}$, and that $\mathbf{p}_n \circ \mathbf{j}_n = \mathrm{id}_{\mathbf{O}^n}$. Therefore $\mathbf{j}_n \in \mathcal{W}$, by Lemma 8. On the other hand $\mathbf{i}_n \in I-\mathrm{cof}$. Since $I-\mathrm{cof}$ is stable under composition and pushout, we have $\mathbf{j}_n \in I-\mathrm{cof}$ $\Diamond$

The next result characterizes the relation of ω-equivalence in terms of suitable factorizations.

Lemma 18. *For any n-cells $x \parallel x'$ in X, the following conditions are equivalent:*

i. $x \sim x'$;

ii. *there is an ω-category Y and ω-functors $k : \partial\mathbf{O}^{n+1} \to Y$, $p : Y \to \mathbf{O}^n$ and $q : Y \to X$ such that $p \in I-\mathrm{inj}$ and the following diagram commutes:*

$$\begin{tikzcd}
 & \partial\mathbf{O}^{n+1} \arrow[dl, "\mathbf{o}_n"'] \arrow[d, dashed, "k"] \arrow[dr, "\langle x,x'\rangle"] & \\
\mathbf{O}^n & Y \arrow[l, dashed, "p"] \arrow[r, dashed, "q"'] & X
\end{tikzcd} ;$$

iii. *There is an ω-functor $q : \mathbf{P}^n \to X$ such that the following diagram commutes:*

$$\begin{tikzcd}
 & \partial\mathbf{O}^{n+1} \arrow[dl, "\mathbf{o}_n"'] \arrow[d, "\mathbf{k}_n"] \arrow[dr, "\langle x,x'\rangle"] & \\
\mathbf{O}^n & \mathbf{P}^n \arrow[l, "\mathbf{p}_n"] \arrow[r, dashed, "q"'] & X
\end{tikzcd} .$$

Proof. If $x \sim x'$, there is a reversible $n{+}1$-cell $u : x \xrightarrow{\sim} x'$ which defines a degenerate n-cylinder $U : x \curvearrowright x'$. We get $\tau_X\, x \parallel U$, whereas $\pi^1_X\, \tau_X\, x = \pi^2_X\, \tau_X\, x = \pi^1_X\, U = x$ and $\pi^2_X\, U = x'$, so that the following diagrams commute:

$$\begin{tikzcd}
\partial\mathbf{O}^{n+1} \arrow[r, "\mathbf{o}_n"] \arrow[d, "\langle\tau_X\, x, U\rangle"'] \arrow[dr, "\langle x,x\rangle"] & \mathbf{O}^n \arrow[d, "\langle x\rangle"] \\
\Gamma(X) \arrow[r, "\pi^1_X"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
\partial\mathbf{O}^{n+1} \arrow[d, "\langle\tau_X\, x, U\rangle"'] \arrow[dr, "\langle x,x'\rangle"] & \\
\Gamma(X) \arrow[r, "\pi^2_X"'] & X
\end{tikzcd}$$

Let $f = \langle x\rangle$. By universality of $\Pi(f)$, we get $k : \partial\mathbf{O}^{n+1} \to \Pi(f)$ such that the following diagram commutes:

$$\begin{tikzcd}
\partial\mathbf{O}^{n+1} \arrow[r, dashed, "k"'] \arrow[rr, bend left, "\mathbf{o}_n"] \arrow[dr, bend right, "\langle\tau_X\, x, U\rangle"'] & \Pi(f) \arrow[r, "f^*\pi^1_X"'] \arrow[d, "f'"'] & \mathbf{O}^n \arrow[d, "f"] \\
 & \Gamma(X) \arrow[r, "\pi^1_X"'] & X
\end{tikzcd}$$

The desired factorizations are given by $Y = \Pi(f)$, $p = f^*\pi^1_X$ and $q = \hat{f} = \pi^2_X \circ f'$. Hence, (i) implies (ii).
Conversely, if we assume (ii), then k gives us two n-cells $y \parallel y'$ in Y such that $p\, y = p\, y'$, $q\, y = x$ and $q\, y' = x'$. Hence, we get $y \sim y'$ by Lemma 6 applied to p, and $x \sim x'$ by Lemma 4 applied to q.
On the other hand, if we assume (ii), then k factors through $\mathbf{k}_n$ by the left lifting property, and so does $\langle x, x'\rangle$. Hence (ii) implies (iii). Conversely, (iii) is just a special case of (ii). $\triangleleft$

We now turn to a new characterization of ω-weak equivalences.

Proposition 8. *An ω-functor $f : X \to Y$ is an ω-weak equivalence if and only if any commutative square whose left arrow is $\mathbf{i}_n$ and whose right arrow is f factors through the generic n-square.*

$$\begin{tikzcd}
\partial\mathbf{O}^n \arrow[r, "\mathbf{i}_n"'] \arrow[d, "\mathbf{i}_n"'] \arrow[rr, bend left] & \mathbf{O}^n \arrow[r, dashed] \arrow[d, "\mathbf{j}_n"] & X \arrow[d, "f"] \\
\mathbf{O}^n \arrow[r, "\mathbf{j}'_n"] \arrow[rr, bend right] & \mathbf{P}^n \arrow[r, dashed] & Y
\end{tikzcd}$$

Proof. Let $f : X \to Y$ be an ω-weak equivalence, and consider a commutative diagram

$$\begin{tikzcd}
\partial\mathbf{O}^n \arrow[r] \arrow[d, "\mathbf{i}_n"'] & X \arrow[d, "f"] \\
\mathbf{O}^n \arrow[r] & Y
\end{tikzcd} .$$

We show that it factors through the generic n-square:

- If $n = 0$, the commutative square is given by some 0-cell y in Y:

$$\begin{tikzcd}
\mathbf{0} \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
\mathbf{1} \arrow[r, "\langle y \rangle"'] & Y
\end{tikzcd}$$

  Since f is in $\mathcal{W}$, there is a 0-cell x in X such that $f\,x \sim y$, and by the previous lemma, we get $q : \mathbf{P}^0 \to Y$ such that $q \circ \mathbf{k}_0 = \langle f\,x, y\rangle$, which means that the following diagram commutes:

$$\begin{tikzcd}
\mathbf{0} \arrow[r] \arrow[d] \arrow[rr, bend left] & \mathbf{1} \arrow[r, dashed, "\langle x \rangle"'] \arrow[d, "\mathbf{j}_0"] & X \arrow[d, "f"] \\
\mathbf{1} \arrow[r, "\mathbf{j}'_0"] \arrow[rr, bend right, "\langle y \rangle"'] & \mathbf{P}^0 \arrow[r, dashed, "q"] & Y
\end{tikzcd}$$

- If $n > 0$, the commutative square is given by $n{-}1$-cells $x \parallel x'$ in X and some n-cell $v : f\,x \to f\,x'$ in Y:

$$\begin{tikzcd}
\partial\mathbf{O}^n \arrow[r, "\langle x, x' \rangle"] \arrow[d, "\mathbf{i}_n"'] & X \arrow[d, "f"] \\
\mathbf{O}^n \arrow[r, "\langle v \rangle"'] & Y
\end{tikzcd}$$

  Since f is in $\mathcal{W}$, there is $u : x \to x'$ in X such that $f\,u \sim v$, and by Lemma 18, we get $q : \mathbf{P}^n \to Y$ such that $q \circ \mathbf{k}_n = \langle f\,u, v\rangle$, which means that the following diagram commutes:

$$\begin{tikzcd}
\partial\mathbf{O}^n \arrow[r, "\mathbf{i}_n"'] \arrow[d, "\mathbf{i}_n"'] \arrow[rr, bend left, "\langle x, x' \rangle"] & \mathbf{O}^n \arrow[r, dashed, "\langle u \rangle"'] \arrow[d, "\mathbf{j}_n"] & X \arrow[d, "f"] \\
\mathbf{O}^n \arrow[r, "\mathbf{j}'_n"] \arrow[rr, bend right, "\langle v \rangle"'] & \mathbf{P}^n \arrow[r, dashed, "q"] & Y
\end{tikzcd}$$

The converse is proved by the same argument. ◁

Corollary 6. *The class $\mathcal{W}$ of ω-weak equivalences admits the solution set $J = \{\mathbf{j}_n | n \in \mathbb{N}\}$.*

We may finally state the central result of this work:

Theorem 3. ω**Cat** *is a combinatorial model category. Its class of weak equivalences is the class $\mathcal{W}$ of ω-weak equivalences while I and J are the sets of generating cofibrations and generating trivial cofibrations, respectively.*

Proof. ω**Cat** is locally presentable by proposition 5 while

- condition **(S1)** holds by lemma 7, lemma 8, lemma 14 and lemma 9;
- condition **(S2)** holds by remark 5;
- condition **(S3)** holds by corollary 1 and corollary 5;
- condition **(S4)** holds by corollary 6. $\triangleleft$

Remark 7. By corollary 3, the model structure of theorem 3 is *left-determined* in the sense of [23].

5 Fibrant and cofibrant objects

Recall that, given a model category **C**, an object X of **C** is *fibrant* if the unique morphism $!_X : X \to 1$ is a fibration. Dually, X is *cofibrant* if the unique morphism $0_X : 0 \to X$ is a cofibration. Now X is fibrant if and only if, for any trivial cofibration $f : Y \to Z$ and any $u : Y \to X$, there is a $v : Z \to X$ such that $v \circ f = u$: in fact, this implies that $!_X : X \to 1$ has the right-lifting property with respect to trivial cofibrations.

$$\begin{tikzcd} Y \arrow[r, "u"] \arrow[d, "f"'] & X \arrow[d, "!_X"] \\ Z \arrow[r, "!_Z"'] \arrow[ur, dashed, "v"] & 1 \end{tikzcd}$$

Likewise, X is cofibrant if and only if for any trivial fibration $p : Y \to Z$ and any morphism $u : X \to Z$ there is a lift $v : X \to Y$ such that $p \circ v = u$.

$$\begin{tikzcd} 0 \arrow[r, "0_Y"] \arrow[d, "0_X"'] & Y \arrow[d, "p"] \\ X \arrow[r, "u"'] \arrow[ur, dashed, "v"] & Z \end{tikzcd}$$

5.1 Fibrant ω-categories

In the folk model structure on ω**Cat**, the characterization of fibrant objects is the simplest possible, as shown by the following result.

Proposition 9. *All ω-categories are fibrant.*

Proof. Let X be an ω-category, $f : Y \to Z$ a trivial cofibration, and $u : Y \to X$ an ω-functor. By Corollary 4, f is an immersion. In particular there is a retraction $g : Z \to Y$ such that $g \circ f = \mathrm{id}_Y$. Let $v = u \circ g$. We get $v \circ f = u \circ g \circ f = u$. Hence X is fibrant.

$$\begin{tikzcd} Y \arrow[r, "u"] \arrow[d, "f"] & X \\ Z \arrow[u, bend left, "g"] \arrow[ur, dashed, "v"'] & \end{tikzcd}$$

$\triangleleft$

5.2 Cofibrant ω-categories

Our understanding of the cofibrant objects in ω**Cat** is based on an appropriate notion of *freely generated* ω-category: notice that the free ω-categories in the sense of the adjunction between ω**Cat** and **Glob** are not sufficient, as there are too few of them. We first describe a process of generating free cells in each dimension. In dimension 0, we just have a set S_0 and no operations, so that S_0 generates $S_0^* = S_0$. In dimension 1, given a graph

$$S_0^* \underset{\tau_0}{\overset{\sigma_0}{\leftleftarrows}} S_1$$

where S_0^* is the set of vertices, S_1 the set of edges, and σ_0, τ_0 are the source and target maps, there is a free category generated by it:

$$S_0^* \underset{\tau_0}{\overset{\sigma_0}{\leftleftarrows}} S_1^* \ .$$

Now suppose that we add a new set S_2 together with a graph

$$S_1^* \underset{\tau_1}{\overset{\sigma_1}{\leftleftarrows}} S_2$$

satisfying the boundary conditions $\sigma_0 \circ \sigma_1 = \sigma_0 \circ \tau_1$ and $\tau_0 \circ \sigma_1 = \tau_0 \circ \tau_1$. What we get is a *computad*, a notion first introduced in [26], freely generating a 2-category

$$S_0^* \underset{\tau_0}{\overset{\sigma_0}{\leftleftarrows}} S_1^* \underset{\tau_1}{\overset{\sigma_1}{\leftleftarrows}} S_2^* \ .$$

This pattern has been extended to all dimensions, giving rise to n-*computads* [21] or *polygraphs* [5, 6]. More precisely, let n**Glob** (resp. n**Cat**) denote the category of n-globular sets (resp. n-categories), we get a commutative diagram

$$\begin{tikzcd} (n{+}1)\mathbf{Cat} \arrow[r] \arrow[d, "U_n"'] & (n{+}1)\mathbf{Glob} \arrow[d] \\ n\mathbf{Cat} \arrow[r] & n\mathbf{Glob} \end{tikzcd} \tag{4}$$

where the horizontal arrows are the obvious forgetful functors and the vertical arrows are truncation functors, removing all $n{+}1$-cells. On the other hand, let $n\mathbf{Cat}^+$ be the category defined by the following pullback square:

$$\begin{tikzcd} n\mathbf{Cat}^+ \arrow[r] \arrow[d, "V_n"'] \arrow[dr, phantom, "\lrcorner", very near start] & (n{+}1)\mathbf{Glob} \arrow[d] \\ n\mathbf{Cat} \arrow[r] & n\mathbf{Glob} \end{tikzcd} \tag{5}$$

From (4), we get a unique functor $R_n : (n{+}1)\mathbf{Cat} \to n\mathbf{Cat}^+$ such that $V_n R_n = U_n$, where U_n and V_n are the truncation functors appearing in (4) and (5) repectively. Now the key to the construction of polygraphs is the existence of a left-adjoint $L_n : n\mathbf{Cat}^+ \to (n{+}1)\mathbf{Cat}$ to this R_n. Concretely, if X is an n-category and S_{n+1} a set of $n{+}1$-cells attached to X by

$$X_n \underset{\tau_n}{\overset{\sigma_n}{\leftleftarrows}} S_{n+1} \tag{6}$$

satisfying the boundary conditions, then L_n builds an $(n{+}1)$-category whose explicit construction is given in [20]. Here we just mention the following property of L_n: let X^+ be an object of $n\mathbf{Cat}^+$ given by an n-category

$$X_0 \underset{\tau_0}{\overset{\sigma_0}{\leftleftarrows}} \cdots \underset{\tau_{n-1}}{\overset{\sigma_{n-1}}{\leftleftarrows}} X_n$$

and a graph (6) then the $n{+}1$-category $L_n X^+$ has the same n-cells as $V_n X^+$. In other words, there is a set of $n{+}1$-cells S_{n+1}^* such that $L_n X^+$ has the form

$$X_0 \underset{\tau_0}{\overset{\sigma_0}{\leftleftarrows}} \cdots \underset{\tau_{n-1}}{\overset{\sigma_{n-1}}{\leftleftarrows}} X_n \underset{\tau_n}{\overset{\sigma_n}{\leftleftarrows}} S_{n+1}^*$$

Definition 16. n-polygraphs *are defined inductively by the following conditions:*

- *a* 0-polygraph *is a set* $S^{(0)}$*;*
- *an* $n{+}1$-polygraph *is an object* $S^{(n+1)}$ *of* $n\mathbf{Cat}^+$ *such that* $V_n S^{(n+1)}$ *is of the form* $L_n S^{(n)}$ *where* $S^{(n)}$ *is an* n*-polygraph.*

Likewise, a polygraph S *is a sequence* $(S^{(n)})_{n\in\mathbb{N}}$ *of* n*-polygraphs such that, for each* n*,* $V_n S^{(n+1)} = L_n S^{(n)}$*.*

The pullback (5) gives a notion of morphisms for $n\mathbf{Cat}^+$, which, by induction, determines a notion of morphism between n-polygraphs, and polygraphs. Thus we get a category **Pol** of polygraphs and morphisms. By Definition 16 and the abovementioned property of L_n, we may see a polygraph S as an infinite diagram of the following shape:

$$\begin{tikzcd}
S_0 \arrow[d] & S_1 \arrow[d] \arrow[dl, shift left] \arrow[dl, shift right] & S_2 \arrow[d] \arrow[dl, shift left] \arrow[dl, shift right] & S_3 \arrow[d] \arrow[dl, shift left] \arrow[dl, shift right] & \cdots . \arrow[dl, shift left] \arrow[dl, shift right] \\
S_0^* & S_1^* \arrow[l, shift left] \arrow[l, shift right] & S_2^* \arrow[l, shift left] \arrow[l, shift right] & S_3^* \arrow[l, shift left] \arrow[l, shift right] & \cdots \arrow[l, shift left] \arrow[l, shift right]
\end{tikzcd} \tag{7}$$

In (7), each S_n is the set of generators of the n-cells, the oblique double arrows represent the attachment of new n-cells on the previously defined $n{-}1$-category, thus defining an object X^+ of $(n{-}1)\mathbf{Cat}^+$, whereas S_n^* is the set of n-cells in $L_{n-1}X^+$. The bottom line of (7) displays the *free ω-category* generated by the polygraph S. This defines a functor $Q : S \mapsto S^*$ from **Pol** to $\omega\mathbf{Cat}$, which is in fact a left-adjoint. A detailed description of the right-adjoint $P : X \mapsto P(X)$ from $\omega\mathbf{Cat}$ to **Pol** is given in [19].

It is now possible to state the main result of this section:

Theorem 4. *An ω-category is cofibrant if and only if it is freely generated by a polygraph.*

Suppose that X is freely generated by a polygraph S, $p : Y \to Z$ is a trivial fibration and $u : X \to Z$ is an ω-functor. It is easy to build a lift $v : X \to Y$ such that $p \circ v = u$ dimensionwise by using the universal property of the functors L_n.

$$\begin{tikzcd}
 & Y \arrow[d, "p"] \\
S^* \arrow[r, "u"'] \arrow[ur, dashed, "v"] & Z
\end{tikzcd}$$

Thus freely generated ω-categories are cofibrant. The proof of the converse is much harder, and is the main purpose of [20]. The problem reduces to the fact that the full subcategory of $\omega\mathbf{Cat}$ whose objects are free on polygraphs is Cauchy complete, meaning that its idempotent morphisms split.

The results of [19] may be revisited in the framework of the folk model structure on $\omega\mathbf{Cat}$. In fact, a resolution of an ω-category X by a polygraph S is a trivial fibration $S^* \to X$, hence a cofibrant replacement of X. Notice that for each ω-category X, the counit of the adjunction between **Pol** and $\omega\mathbf{Cat}$ gives an ω-functor

$$\epsilon_X : QP(X) \to X$$

which is a trivial fibration, and defines the *standard resolution* of X.

6 Model structure on nCat

In this section, we show that the model structure on $\omega\mathbf{Cat}$ we just described yields a model structure on the category $n\mathbf{Cat}$ of (strict, small) n-categories for each integer $n \geq 1$. In particular, we recover the known folk model structures on **Cat** [13] and $2\mathbf{Cat}$ [15, 16].

Let $n \geq 1$ be a fixed integer. There is an inclusion functor

$$F : n\mathbf{Cat} \to \omega\mathbf{Cat}$$

which simply adds all necessary unit cells in dimensions $k > n$. This functor F has a left adjoint

$$G : \omega\mathbf{Cat} \to n\mathbf{Cat}.$$

Precisely, if X is an ω-category and $0 \leq k \leq n$, the k-cells of GX are exactly those of X for $k < n$, whereas $(GX)_n$ is the quotient of X_n modulo the *congruence generated* by X_{n+1}. In other words, parallel n-cells x, y in X are *congruent modulo* X_{n+1} if and only if there is a sequence $x_0 = x, x_1, \ldots, x_p = y$ of n-cells and a sequence $z_1, \ldots, z_p$ of n+1-cells such that, for each $i = 1, \ldots, p$ either $z_i : x_{i-1} \to_n x_i$ or $z_i : x_i \to_n x_{i-1}$.

Notice that the functor F also has a right adjoint, namely the *truncation* functor $U : \omega\mathbf{Cat} \to n\mathbf{Cat}$ which simply forgets all cells of dimension $k > n$.

Theorem 5. *The inclusion functor $F : n\mathbf{Cat} \to \omega\mathbf{Cat}$ creates a model structure on $n\mathbf{Cat}$, in which the weak equivalences are the n-functors f such that $F(f) \in \mathcal{W}$, and $(G(\mathbf{i}_k))_{k \in \mathbb{N}}$ is a family of generating cofibrations.*

The general situation is investigated in [4], whose proposition 2.3 states sufficient conditions for the transport of a model structure along an adjunction. In our particular case, these conditions boil down to the following:

(C1) the model structure on $\omega\mathbf{Cat}$ is cofibrantly generated;

(C2) $n\mathbf{Cat}$ is locally presentable;

(C3) $\mathcal{W}$ is closed under filtered colimits in $\omega\mathbf{Cat}$;

(C4) F preserves filtered colimits;

(C5) If $j \in J$ is a generating trivial cofibration of $\omega\mathbf{Cat}$, and g is a pushout of $G(j)$ in $n\mathbf{Cat}$, then $F(g)$ is a weak equivalence in $\omega\mathbf{Cat}$.

Conditions **(C1)** and **(C2)** are known already. Condition **(C3)** follows from the definition of weak equivalences and the fact that the ω-categories $\mathbf{O}^n$ are finitely presentable objects in $\omega\mathbf{Cat}$. The functor F, being left adjoint to U, preserves all colimits, in particular filtered ones, hence **(C4)** .

We now turn to the proof of the remaining condition **(C5)** . First remark that GF is the identity on $n\mathbf{Cat}$, so that the monad $T = FG$ is idempotent and the monad multiplication $\mu : T^2 \to T$ is the identity. As a consequence, if $\eta : 1 \to T$ denotes the unit of the monad, for each ω-category X

$$T(\eta_X) = 1_{T(X)}. \tag{8}$$

Also, for each ω-functor of the form $u : T(X) \to T(Y)$,

$$T(u) = u. \tag{9}$$

Now let X be an ω-category. For each $k > n$, all k-cells of $T(X)$ are units. Therefore, by construction of the connection functor Γ, all k-cells in $\Gamma T(X)$ are also units, which implies that $\Gamma T(X)$ belongs to the image of F, whence

$$T\Gamma T(X) = \Gamma T(X). \tag{10}$$

We successively get the natural transformations:

$$\eta_X : X \to T(X), \qquad \Gamma(\eta_X) : \Gamma(X) \to \Gamma T(X), \qquad T\Gamma(\eta_X) : T\Gamma(X) \to T\Gamma T(X) = \Gamma T(X),$$

by (10). Thus $\lambda_X = T\Gamma(\eta_X)$ yields a natural transformation

$$\lambda : T\Gamma \to \Gamma T.$$

Lemma 19. *The monad T on ω**Cat** preserves immersions.*

Proof. Let $f : X \to Y$ be an immersion. We want to show that $f' = T(f)$ is still an immersion. By Definition 14, there are $g : Y \to X$ and $h : Y \to \Gamma(Y)$ such that

$$\begin{aligned} g \circ f &= \mathrm{id}; \\ \pi^1_Y \circ h &= f \circ g; \\ \pi^2_Y \circ h &= \mathrm{id}; \\ h \circ f &= \tau_Y \circ f. \end{aligned}$$

Let $g' = T(g) : T(Y) \to T(X)$ and $h' = \lambda_Y \circ T(h) : T(Y) \to \Gamma T(Y)$, it is now sufficient to check the following equations:

$$g' \circ f' = \mathrm{id}; \tag{11}$$
$$\pi^1_{T(Y)} \circ h' = f' \circ g'; \tag{12}$$
$$\pi^2_{T(Y)} \circ h' = \mathrm{id}; \tag{13}$$
$$h' \circ f' = \tau_{T(Y)} \circ f'. \tag{14}$$

Equation (11) is obvious from functoriality. As for (12), we first notice that, by naturality of π^1, the following diagram commutes:

$$\begin{tikzcd} \Gamma(Y) \arrow[r, "\pi^1_Y"] \arrow[d, "\Gamma(\eta_Y)"'] & Y \arrow[d, "\eta_Y"] \\ \Gamma T(Y) \arrow[r, "\pi^1_{T(Y)}"'] & T(Y) \end{tikzcd} .$$

By applying T to the above diagram, we get

$$\begin{tikzcd} T\Gamma(Y) \arrow[r, "T(\pi^1_Y)"] \arrow[d, "\lambda_Y"'] & T(Y) \arrow[d, "T(\eta_Y)"] \\ \Gamma T(Y) \arrow[r, "T(\pi^1_{T(Y)})"'] & T(Y) \end{tikzcd} .$$

Now, by (8), $T(\eta_Y) = 1_{T(Y)}$ and because $\Gamma T(Y) = T\Gamma T(Y)$, by (9), $T(\pi^1_{T(Y)}) = \pi^1_{T(Y)}$. Hence

$$T(\pi^1_Y) = \pi^1_{T(Y)} \circ \lambda_Y. \tag{15}$$

Thus

$$\begin{aligned} \pi^1_{T(Y)} \circ h' &= \pi^1_{T(Y)} \circ \lambda_Y \circ T(h), \\ &= T(\pi^1_Y) \circ T(h), \\ &= T(\pi^1_Y \circ h), \\ &= T(f \circ g), \\ &= T(f) \circ T(g), \\ &= f' \circ g'. \end{aligned}$$

Equations (13) and (14) hold by the same arguments applied to the natural transformations π^2 and τ respectively. Hence $T(f)$ is an immersion, and we are done. $\triangleleft$

Lemma 20. *Let $f : X \to Y$ be an immersion, and suppose the following square is a pushout in n**Cat**:*

$$\begin{tikzcd} G(X) \arrow[r, "u"] \arrow[d, "G(f)"'] & A \arrow[d, "g"] \\ G(Y) \arrow[r, "v"'] \arrow[ur, phantom, "\ulcorner"] & B \end{tikzcd}$$

Then $F(g)$ is an immersion.

Proof. As F is left adjoint to U, it preserves pushouts, and the following square is a pushout in ω**Cat**:

$$\begin{tikzcd} T(X) \arrow[r, "F(u)"] \arrow[d, "T(f)"'] & F(A) \arrow[d, "F(g)"] \\ T(Y) \arrow[r, "F(v)"'] & F(B) \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Now f is an immersion, and so is $T(f)$ by Lemma 19. As immersions are closed by pushouts (Lemma 17), $F(g)$ is also an immersion. ◁

Now let j be a generating trivial cofibration in ω**Cat**, and g a pushout of $G(j)$ in n**Cat**. By Corollary 4, j is an immersion, so that Lemma 20 applies, and $F(g)$ is an immersion. By Lemma 16, immersions are weak equivalences, so that $F(g) \in \mathcal{W}$. Hence condition **(C5)** holds, and we are done.

In case $n = 1$, the weak equivalences of n**Cat** are exactly the equivalences of categories, whereas if $n = 2$, they are the *biequivalences* in the sense of [15]. Moreover, from the generating cofibrations of ω**Cat** we immediately get a family of generating cofibrations in n**Cat**, namely the n-functors

$$G(\mathbf{i}_k) : G(\partial\mathbf{O}^k) \to G(\mathbf{O}^k)$$

for all $k \in \mathbb{N}$. By abuse of language, let us denote $G(X) = X$ whenever X is an ω-category of the form $F(Y)$, that is without non-identity cells in dimensions $> n$. Likewise, denote $G(f) = f$ for each ω-functor f of the form $F(g)$. With this convention

– for each integer $k \leq n$, $G(\mathbf{i}_k) = \mathbf{i}_k$;

– $G(\mathbf{i}_{n+1})$ is the collapsing map $\mathbf{i}'_{n+1} : \partial\mathbf{O}^{n+1} \to \mathbf{O}^n$;

– for each $k > n+1$, $G(\mathbf{i}_k)$ is the identity on $\mathbf{O}^n$.

Now the right-lifting property with respect to identities is clearly void. Thus we only need a *finite* family of $n+2$ generating cofibrations

$$\mathbf{i}_0, \ldots, \mathbf{i}_n, \mathbf{i}'_{n+1}.$$

If $n = 1$ or $n = 2$, these are precisely the generating cofibrations of [13] and [15] respectively. Therefore the corresponding model structures are particular cases of ours.

A The functor Γ

The aim of this section is to give a complete proof of Theorem 2. In order to do that, we extend ω-functors to cylinders and we introduce the following operations:

– *left* and *right action* of cells on cylinders, written $u \cdot V$ and $U \cdot v$;

– *concatenation* of cylinders, written $U * V$;

– *multiplication* of cylinders, written $U \circledast V$;

– *compositions* of cylinders and the *units*, written $U *_n V$ and 1_U^m.

We must prove the following properties: associativity and units for compositions, interchange and iterated units, compatibility of $\Gamma(f), \pi^1, \pi^2, \tau$ with compositions and units, functoriality of Γ and naturality of π^1, π^2, τ.

Lemma 21. *(functoriality) Any ω-functor $f : X \to Y$ extends to cylinders in a canonical way:*

i. for any n-cylinder $U : x \curvearrowright x'$ in X, we get some n-cylinder $f\,U : f\,x \curvearrowright f\,x'$ in Y;

ii. *we have* $f\,U \parallel f\,V$ *whenever* $U \parallel V$*, and* $f\,W : f\,U \to f\,V$ *for any* $W : U \to V$*;*

iii. *we have* $(g \circ f)\,U = g\,f\,U$ *for any* ω*-functor* $g : Y \to Z$*, and also* $\mathrm{id}\,U = U$*.*

In other words, Γ *defines a functor from* ω**Cat** *to* **Glob** *and the homomorphisms* π^1, π^2 *are natural.*

Definition 17. *(left and right action) Precomposition and postcomposition extend to cylinders. For any 0-cells* x, y, z*, we get:*

- *the* n*-cylinder* $u \cdot V$ *in* $[x, z]$*, defined for any 1-cell* $u : x \to y$ *and for any* n*-cylinder* V *in* $[y, z]$*;*
- *the* n*-cylinder* $U \cdot v$ *in* $[x, z]$*, defined for any 1-cell* $v : y \to z$ *and for any* n*-cylinder* U *in* $[x, y]$*.*

Lemma 22. *(bimodularity) The following identities hold for any 0-cells* x, y, z, t*:*

- $(u *_0 v) \cdot W = u \cdot (v \cdot W)$ *for any 1-cells* $u : x \to y$ *and* $v : y \to z$*, and for any* n*-cylinder* W *in* $[z, t]$*;*
- $(U \cdot v) \cdot w = U \cdot (v *_0 w)$ *for any 1-cells* $v : y \to z$ *and* $w : z \to t$*, and for any* n*-cylinder* U *in* $[x, y]$*;*
- $(u \cdot V) \cdot w = u \cdot (V \cdot w)$ *for any 1-cells* $u : x \to y$ *and* $w : z \to t$*, and for any* n*-cylinder* V *in* $[y, z]$*.*

Moreover, we have $1_x \cdot U = U = U \cdot 1_y$ *for any 0-cells* x, y *and for any* n*-cylinder* U *in* $[x, y]$*.*

This is proved by functoriality.
We omit parentheses in such expressions: For instance, $u \cdot v \cdot W$ stands for $u \cdot (v \cdot W)$, and $U \cdot v \cdot w$ for $(U \cdot v) \cdot w$. Moreover, action will always have precedence over other operations: For instance, $u \cdot V * W$ stands for $(u \cdot V) * W$.

Definition 18. *(concatenation) By induction on* n*, we define the* n*-cylinder* $U * V : x \curvearrowright z$ *for any* n*-cylinders* $U : x \curvearrowright y$ *and* $V : y \curvearrowright z$*:*

- *if* $n = 0$*, then* $(U * V)^\natural = U^\natural *_0 V^\natural$*;*
- *if* $n > 0$*, then* $(U * V)^\flat = U^\flat *_0 V^\flat$ *and* $(U * V)^\sharp = U^\sharp *_0 V^\sharp$*, whereas* $[U * V] = [U] \cdot V^\sharp * U^\flat \cdot [V]$*.*

In both cases, we say that U *and* V *are* consecutive*, and we write* $U \triangleright V$*.*

Lemma 23. *(source and target of a concatenation) We have* $U * U' \parallel V * V'$ *for any* n*-cylinders* $U \parallel V$ *and* $U' \parallel V'$ *such that* $U \triangleright U'$ *and* $V \triangleright V'$*, and* $W * W' : U * U' \to V * V'$ *for any* $n{+}1$*-cylinders* $W : U \to V$ *and* $W' : U' \to V'$ *such that* $W \triangleright W'$*.*

Lemma 24. *(compatibility of* $\Gamma(f)$ *with concatenation and* τ*) The following identities hold any* ω*-functor* $f : X \to Y$*:*

- $f(U * V) = f\,U * f\,V$ *for any* n*-cylinders* $U \triangleright V$ *in* X*;*
- $f\,\tau\,x = \tau\,f\,x$ *for any* n*-cell* x *in* X*.*

In particular, the homomorphism τ *is natural.*

In the cases of precomposition and postcomposition, we get the following result:

Lemma 25. *(distributivity over concatenation and* τ*) The following identities hold for any 0-cells* x, y, z *and for any 1-cell* $u : x \to y$*:*

- $u \cdot (V * W) = u \cdot V * u \cdot W$ *for any* n*-cylinders* $V \triangleright W$ *in* $[y, z]$*;*
- $u \cdot \tau[v] = \tau[u *_0 v]$ *for any* $n{+}1$*-cell* $v : y \to_0 z$*.*

There are similar properties for right action.

Lemma 26. *(associativity and units for concatenation) The following identities hold for any* n*-cylinders* $U \triangleright V \triangleright W$ *and for any* n*-cylinder* $U : x \curvearrowright y$*:*

$$(U * V) * W = U * (V * W), \qquad \tau\,x * U = U = U * \tau\,y.$$

Proof. We proceed by induction on n.
The case $n = 0$ is obvious.
If $n > 0$, the first identity is obtained as follows:

$$\begin{aligned}
[(U * V) * W] &= [U * V] \cdot W^\sharp * (U * V)^\flat \cdot [W] && \text{(definition of } *\text{)}\\
&= ([U] \cdot V^\sharp * U^\flat \cdot [V]) \cdot W^\sharp * (U^\flat *_0 V^\flat) \cdot [W] && \text{(definition of } *\text{)}\\
&= ([U] \cdot V^\sharp \cdot W^\sharp * U^\flat \cdot [V] \cdot W^\sharp) * U^\flat \cdot V^\flat \cdot [W] && \text{(distributivity over } *\text{)}\\
&= [U] \cdot V^\sharp \cdot W^\sharp * (U^\flat \cdot [V] \cdot W^\sharp * U^\flat \cdot V^\flat \cdot [W]) && \text{(induction hypothesis)}\\
&= [U] \cdot (V^\sharp *_0 W^\sharp) * U^\flat \cdot ([V] \cdot W^\sharp * V^\flat \cdot [W]) && \text{(distributivity over } *\text{)}\\
&= [U] \cdot (V * W)^\sharp * U^\flat \cdot [V * W] && \text{(definition of } *\text{)}\\
&= [U * (V * W)]. && \text{(definition of } *\text{)}
\end{aligned}$$

The second identity is obtained as follows, using distributivity over τ and the induction hypothesis:

$$[\tau\, x * U] = [\tau\, x] \cdot U^\sharp * (\tau\, x)^\flat \cdot [U] = \tau[x] \cdot U^\sharp * 1_{x^\flat} \cdot [U] = \tau\big[x *_0 U^\sharp\big] * [U] = [U],$$

and similarly for the third one. ◁

From now on, we shall omit parentheses in concatenations.

Lemma 27. *(cylinders in a cartesian product) There are natural isomorphisms of globular sets* $\Gamma(X \times Y) \simeq \Gamma(X) \times \Gamma(Y)$ *and* $\Gamma(\mathbf{1}) \simeq \mathbf{1}$*, which satisfy the following coherence conditions with the canonical isomorphisms* $(X \times Y) \times Z \simeq X \times (Y \times Z)$ *and* $\mathbf{1} \times X \simeq X \simeq X \times \mathbf{1}$*:*

$$\begin{tikzcd}
\Gamma((X \times Y) \times Z) \arrow[r] \arrow[d] & \Gamma(X \times (Y \times Z)) \arrow[d] \\
\Gamma(X \times Y) \times \Gamma(Z) \arrow[d] & \Gamma(X) \times \Gamma(Y \times Z) \arrow[d] \\
(\Gamma(X) \times \Gamma(Y)) \times \Gamma(Z) \arrow[r] & \Gamma(X) \times (\Gamma(Y) \times \Gamma(Z))
\end{tikzcd}
\qquad
\begin{tikzcd}
\Gamma(\mathbf{1} \times X) \arrow[r] \arrow[d] & \Gamma(X) \arrow[dd, equal] & \Gamma(X \times \mathbf{1}) \arrow[l] \arrow[d] \\
\Gamma(\mathbf{1}) \times \Gamma(X) \arrow[d] & & \Gamma(X) \times \Gamma(\mathbf{1}) \arrow[d] \\
\mathbf{1} \times \Gamma(X) \arrow[r] & \Gamma(X) & \Gamma(X) \times \mathbf{1} \arrow[l]
\end{tikzcd}$$

Remark 8. There is a coherence condition for the symmetry $X \times Y \simeq Y \times X$, but we shall not use it explicitly. ♢

Remark 9. By Lemmas 21 and 27, any ω-bifunctor $f : X \times Y \to Z$ extends to cylinders in a canonical way. ♢

Definition 19. *(multiplication) Composition extends to cylinders: For any 0-cells* x, y, z*, we get the* n*-cylinder* $U \circledast V$ *in* $[x, z]$*, defined for any* n*-cylinders* U *in* $[x, y]$ *and* V *in* $[y, z]$*.*

Lemma 28. *(associativity of multiplication) The following identity holds for any 0-cells* x, y, z, t*, and for any* n*-cylinders* U *in* $[x, y]$*,* V *in* $[y, z]$*,* W *in* $[z, t]$*:*

$$(U \circledast V) \circledast W = U \circledast (V \circledast W)$$

Proof. By functoriality, using coherence with the canonical isomorphism $(X \times Y) \times Z \simeq X \times (Y \times Z)$. ◁

Remark 10. In $\Gamma(X \times Y) \simeq \Gamma(X) \times \Gamma(Y)$, concatenation and τ can be defined componentwise. ♢

Using compatibility of $\Gamma(f)$ with concatenation and τ, we get the following result:

Lemma 29. *(compatibility of multiplication with concatenation and* τ*) The following identities hold for any 0-cells* x, y, z*, for any* n*-cylinders* $U \rhd U'$ *in* $[x, y]$ *and* $V \rhd V'$ *in* $[y, z]$*, and for any* $n{+}1$*-cells* $u : x \to_0 y$ *and* $v : y \to_0 z$*:*

$$(U * U') \circledast (V * V') = (U \circledast V) * (U' \circledast V'), \qquad \tau[u] \circledast \tau[v] = \tau[u *_0 v].$$

Remark 11. Any 0-cell x in X defines an ω-functor $\langle x \rangle : \mathbf{1} \to X$, from which we get $\Gamma\langle x \rangle : \mathbf{1} \simeq \Gamma(\mathbf{1}) \to \Gamma(X)$. It is easy to see that this homomorphism of globular sets corresponds to the sequence of trivial n-cylinders $\tau\, 1_x^n$. ♢

Lemma 30. *(representability) The following identities hold for any 0-cells* x, y, z*:*

– $u \cdot V = \tau\, 1_{[u]}^n \circledast V = \tau\big[1_u^{n+1}\big] \circledast V$ *for any 1-cell* $u : x \to y$ *and for any* n*-cylinder* V *in* $[y, z]$*;*

– $U \cdot v = U \circledast \tau\, 1^n_{[v]} = U \circledast \tau\big[1^{n+1}_v\big]$ *for any 1-cell* $v : y \to z$ *and for any* n*-cylinder* U *in* $[x, y]$.

In other words, the (left and right) action of a 1-cell u *is represented by the* n*-cylinder* $\tau\big[1^{n+1}_u\big]$.

Proof. By functoriality, using coherence with the canonical isomorphisms $\mathbf{1} \times X \simeq X \simeq X \times \mathbf{1}$. ◁

Definition 20. *(extended action) For any 0-cells* x, y, z*, we extend left and right action to higher dimensional cells as follows:*

– $u \cdot V = \tau[u] \circledast V$ *for any* $n{+}1$*-cell* $u : x \to y$ *and for any* n*-cylinder* V *in* $[y, z]$*;*

– $U \cdot v = U \circledast \tau[v]$ *for any* $n{+}1$*-cell* $v : y \to z$ *and for any* n*-cylinder* U *in* $[x, y]$.

Remark 12. In particular, we get $u \cdot V = 1^{n+1}_u \cdot V$ for any 1-cell $u : x \to y$ and for any n-cylinder V in $[y, z]$, and similarly for the right action. This means that we have indeed extended the action of 1-cells. ♢

Lemma 31. *(extended bimodularity) The first three identities of lemma 22 extend to higher dimensional cells.*

Proof. By associativity of multiplication and compatibility of multiplication with τ. ◁

Lemma 32. *(extended distributivity) The identities of lemma 25 extend to higher dimensional cells.*

Proof. The first identity is obtained as follows, using compatibility of multiplication with concatenation:

$$u \cdot (V * W) = \tau[u] \circledast (V * W) = (\tau[u] * \tau[u]) \circledast (V * W) = (\tau[u] \circledast V) * (\tau[u] \circledast W) = u \cdot V * u \cdot W.$$

The second one follows from compatibility of multiplication with τ. ◁

Lemma 33. *(commutation) The following identities hold for any 0-cells* x, y, z*, for any* $n{+}1$*-cells* $u, u' : x \to_0 y$ *and* $v, v' : y \to_0 z$*, and for any* n*-cylinders* $U : [u] \curvearrowright [u']$ *in* $[x, y]$ *and* $V : [v] \curvearrowright [v']$ *in* $[y, z]$*:*

$$U \cdot v * u' \cdot V = U \circledast V = u \cdot V * U \cdot v'.$$

Proof. The first identity is obtained as follows, using compatibility of multiplication with concatenation:

$$U \cdot v * u' \cdot V = (U \circledast \tau[v]) * (\tau[u'] \circledast V) = (U * \tau[u']) \circledast (\tau[v] * V) = U \circledast V,$$

and similarly for the second one. ◁

From now on, we shall always assume that $m > n$.

Definition 21. *(compositions) By induction on* n*, we define the* m*-cylinder* $U *_n V : R \to_n T \mid x *_n y \curvearrowright x' *_n y'$ *for any* m*-cylinders* $U : R \to_n S \mid x \curvearrowright x'$ *and* $V : S \to_n T \mid y \curvearrowright y'$*:*

– $(U *_0 V)^\flat = U^\flat = R^\natural$ *and* $(U *_0 V)^\sharp = V^\sharp = T^\natural$*, whereas* $[U *_0 V] = x \cdot [V] * [U] \cdot y'$*;*

– *if* $n > 0$*, then* $(U *_n V)^\flat = U^\flat = V^\flat$ *and* $(U *_n V)^\sharp = U^\sharp = V^\sharp$*, whereas* $[U *_n V] = [U] *_{n-1} [V]$.

In both cases, we say that U *and* V *are* n-composable*, and we write* $U \rhd_n V$.

Lemma 34. *(source and target of a composition) We have* $U *_n U' \parallel V *_n V'$ *for any* m*-cylinders* $U \parallel V$ *and* $U' \parallel V'$ *such that* $U \rhd_n U'$ *(so that* $V \rhd_n V'$*), and* $W *_n W' : U *_n U' \to V *_n V'$ *for any* $m{+}1$*-cylinders* $W : U \to V$ *and* $W' : U' \to V'$.

Definition 22. *(units) By induction on* n*, we define the* m*-cylinder* $1^m_U : U \to_n U \mid 1^m_x \curvearrowright 1^m_y$ *for any* n*-cylinder* $U : x \curvearrowright y$*:*

– *if* $n = 0$*, then* $(1^m_U)^\flat = (1^m_U)^\sharp = U^\natural$*, whereas* $[1^m_U] = \tau\big[1^m_{U^\natural}\big]$*. In particular, we get* $\big[1^1_U\big] = \tau\big[U^\natural\big]$*;*

– *if* $n > 0$*, then* $(1^m_U)^\flat = U^\flat$ *and* $(1^m_U)^\sharp = U^\sharp$*, whereas* $[1^m_U] = 1^{m-1}_{[U]}$.

Lemma 35. *(source and target of a unit) We have* $1^{m+1}_U : 1^m_U \to 1^m_U$ *for any* n*-cylinder* U.

Remark 13. By construction, π^1 and π^2 are compatible with compositions and units. ♢

Lemma 36. *(associativity and units for compositions) The following identities hold for any m-cylinders $U \rhd_n V \rhd_n W$ and for any m-cylinder $U : S \to_n T$:*

$$(U *_n V) *_n W = U *_n (V *_n W), \qquad 1^m_S *_n U = U = U *_n 1^m_T.$$

Proof. We proceed by induction on n.
If $n = 0$, the first identity is obtained as follows (with $U : x \curvearrowright x'$, $V : y \curvearrowright y'$ and $W : z \curvearrowright z'$):

$$\begin{aligned}
[(U *_0 V) *_0 W] &= (x *_0 y) \cdot [W] * [U *_0 V] \cdot z' && \text{(definition of } *_0\text{)}\\
&= x \cdot y \cdot [W] * (x \cdot [V] * [U] \cdot y') \cdot z' && \text{(definition of } *_0\text{)}\\
&= x \cdot y \cdot [W] * x \cdot [V] \cdot z' * [U] \cdot y' \cdot z' && \text{(distributivity over } *\text{)}\\
&= x \cdot (y \cdot [W] * [V] \cdot z') * [U] \cdot y' \cdot z' && \text{(distributivity over } *\text{)}\\
&= x \cdot [V * W] * [U] \cdot (y' *_0 z') && \text{(definition of } *_0\text{)}\\
&= [U *_0 (V *_0 W)]. && \text{(definition of } *_0\text{)}
\end{aligned}$$

The second identity is obtained as follows (with $U : x \curvearrowright y$ and $S : x^\flat \curvearrowright y^\flat$), using distributivity over τ:

$$[1^m_S *_0 U] = 1^m_{x^\flat} \cdot [U] * [1^m_S] \cdot y = 1_{x^\flat} \cdot [U] * \tau[1^m_{S^\natural}] \cdot y = [U] * \tau[1^m_{S^\natural} *_0 y] = [U],$$

and similarly for the third one.
If $n > 0$, we apply the induction hypothesis. $\lhd$

Lemma 37. *(compatibility of τ with compositions and units) The following identities hold for any m-cells $u \rhd_n v$ and for any n-cell x:*

$$\tau(u *_n v) = \tau\, u *_n \tau\, v, \qquad \tau\, 1^m_x = 1^m_{\tau\, x}.$$

Proof. By induction on n.
If $n = 0$, the first identity is obtained as follows, using distributivity over τ:

$$[\tau(u *_0 v)] = \tau[u *_0 v] = \tau[u *_0 v] * \tau[u *_0 v] = u \cdot \tau[v] * \tau[u] \cdot v = u \cdot [\tau\, v] * [\tau\, u] \cdot v = [\tau\, u *_0 \tau\, v].$$

The second identity is obtained as follows:

$$[\tau\, 1^m_x] = \tau[1^m_x] = \tau\left[1^m_{1_x}\right] = \tau\left[1^m_{(\tau\, x)^\natural}\right] = [1^m_{\tau\, x}].$$

If $n > 0$, we apply the induction hypothesis. $\lhd$

Lemma 38. *(compatibility of $\Gamma(f)$ with compositions and units) The following identities hold any ω-functor $f : X \to Y$:*

- *$f(U *_n V) = f\, U *_n f\, V$ for any m-cylinders $U \rhd_n V$ in X;*
- *$f\, 1^m_U = 1^m_{f\, U}$ for any n-cylinder U in X.*

In the cases of precomposition and postcomposition, we get the following result:

Lemma 39. *(distributivity over compositions and units) The following identities for any 0-cells x, y, z and for any 1-cell $u : x \to y$:*

- *$u \cdot (V *_n W) = u \cdot V *_n u \cdot W$ for any m-cylinders $V \rhd_n W$ in $[y, z]$;*
- *$u \cdot 1^m_V = 1^m_{u \cdot V}$ for any n-cylinder V in $[y, z]$.*

There are similar properties for right action.

Lemma 40. *(compatibility of concatenation with composition and units) The following identities hold for any m-cylinders $U \rhd_n V$ and $U' \rhd_n V'$ such that $U \rhd U'$ and $V \rhd V'$, and for any n-cylinders $S \rhd T$:*

$$(U *_n V) * (U' *_n V') = (U * U') *_n (V * V'), \qquad 1^m_S * 1^m_T = 1^m_{S * T}.$$

Proof. We proceed by induction on n.
If $n=0$, the first identity is obtained as follows (with $U : x \curvearrowright x'$, $U' : x' \curvearrowright x''$, $V : y \curvearrowright y'$ and $V' : y' \curvearrowright y''$):

$$\begin{aligned}
[(U *_0 V) * (U' *_0 V')] &= [U *_0 V] \cdot (U' *_0 V')^\sharp * (U *_0 V)^\flat \cdot [U' *_0 V'] && \text{(definition of } *) \\
&= (x \cdot [V] * [U] \cdot y') \cdot V'^\sharp * U^\flat \cdot (x' \cdot [V'] * [U'] \cdot y'') && \text{(definition of } *_0) \\
&= x \cdot [V] \cdot V'^\sharp * [U] \cdot y' \cdot V'^\sharp * U^\flat \cdot x' \cdot [V'] * U^\flat \cdot [U'] \cdot y'' && \text{(distributivity over } *) \\
&= x \cdot [V] \cdot V'^\sharp * x \cdot V^\flat \cdot [V'] * [U] \cdot U'^\sharp \cdot y'' * U^\flat \cdot [U'] \cdot y'' && \text{(commutation)} \\
&= x \cdot ([V] \cdot V'^\sharp * V^\flat \cdot [V']) * ([U] \cdot U'^\sharp * U^\flat \cdot [U']) \cdot y'' && \text{(distributivity over } *) \\
&= x \cdot [V * V'] * [U * U'] \cdot y'' && \text{(definition of } *) \\
&= [(U * U') *_0 (V * V')]. && \text{(definition of } *_0)
\end{aligned}$$

In the commutation step, we use the fact that $U^\sharp = V^\flat$ and $U'^\sharp = V'^\flat$ since $U \rhd_0 V$ and $U' \rhd_0 V'$.
The second identity is obtained as follows, using distributivity over τ:

$$[1_S^m * 1_T^m] = [1_S^m] \cdot (1_T^m)^\sharp * (1_S^m)^\flat \cdot [1_T^m] = \tau\big[1_{S^\natural}^m\big] \cdot T^\natural * S^\natural \cdot \tau\big[1_{T^\natural}^m\big] =$$
$$\tau\Big[1_{S^\natural *_0 T^\natural}^m\Big] * \tau\Big[1_{S^\natural *_0 T^\natural}^m\Big] = \tau\Big[1_{S^\natural *_0 T^\natural}^m\Big] = \tau\Big[1_{(S*T)^\natural}^m\Big] = [1_{S*T}^m].$$

If $n>0$, the first identity is obtained as follows:

$$\begin{aligned}
[(U *_n V) * (U' *_n V')] &= [U *_n V] \cdot (U' *_n V')^\sharp * (U *_n V)^\flat \cdot [U' *_n V'] && \text{(definition of } *) \\
&= ([U] *_{n-1} [V]) \cdot U'^\sharp * U^\flat \cdot ([U'] *_{n-1} [V']) && \text{(definition of } *_n) \\
&= ([U] \cdot U'^\sharp *_{n-1} [V] \cdot U'^\sharp) * (U^\flat \cdot [U'] *_{n-1} U^\flat \cdot [V']) && \text{(distributivity over } *_{n-1}) \\
&= ([U] \cdot U'^\sharp * U^\flat \cdot [U']) *_{n-1} ([V] \cdot U'^\sharp * U^\flat \cdot [V']) && \text{(induction hypothesis)} \\
&= [U * U'] *_{n-1} [V * V'] && \text{(definition of } *) \\
&= [(U * U') *_n (V * V')]. && \text{(definition of } *_n)
\end{aligned}$$

In the penultimate step, we use the fact that $U^\flat = V^\flat$ and $U'^\sharp = V'^\sharp$ since $U \rhd_n V$ and $U' \rhd_n V'$.
The second identity is obtained as follows, using distributivity over units and the induction hypothesis:

$$[1_S^m * 1_T^m] = [1_S^m] \cdot (1_T^m)^\sharp * (1_S^m)^\flat \cdot [1_T^m] = 1_{[S]}^{m-1} \cdot T^\sharp * S^\flat \cdot 1_{[T]}^{m-1} =$$
$$1_{[S]\cdot T^\sharp}^{m-1} * 1_{S^\flat \cdot [T]}^{m-1} = 1_{[S]\cdot T^\sharp * S^\flat \cdot [T]}^{m-1} = 1_{[S*T]}^{m-1} = [1_{S*T}^m]. \qquad \triangleleft$$

Remark 14. In $\Gamma(X \times Y) \simeq \Gamma(X) \times \Gamma(Y)$, compositions and units can be defined componentwise. $\Diamond$

Using compatibility of $\Gamma(f)$ with compositions and units, we get the following result:

Lemma 41. *(compatibility of multiplication with compositions and units) The following identities hold for any 0-cells x, y, z, for any m-cylinders $U \rhd_n U'$ in $[x, y]$ and $V \rhd_n V'$ in $[y, z]$, and for any n-cylinders S in $[x, y]$ and T in $[y, z]$:*

$$(U *_n U') \circledast (V *_n V') = (U \circledast V) *_n (U' \circledast V'), \qquad 1_S^m \circledast 1_T^m = 1_{S \circledast T}^m.$$

Lemma 42. *(compatibility of action with compositions and units) The following identities hold for any 0-cells x, y, z, for any $m+1$-cells $u, u' : x \to_0 y$ such that $u \rhd_{n+1} u'$, for any m-cylinders $V \rhd_n V'$ in $[y, z]$, for any $n+1$-cell $s : x \to_0 y$, and for any n-cylinder T in $[y, z]$:*

$$(u *_{n+1} u') \cdot (V *_n V') = u \cdot V *_n u' \cdot V', \qquad 1_s^{m+1} \cdot 1_T^m = 1_{s \cdot T}^m.$$

There are similar properties for right action.

Proof. The first identity is obtained as follows, using compatibility of τ with compositions and the previous lemma:

$$(u *_{n+1} u') \cdot (V *_n V') = \tau[u *_{n+1} u'] \circledast (V *_n V') = \tau([u] *_n [u']) \circledast (V *_n V') =$$
$$(\tau[u] *_n \tau[u']) \circledast (V *_n V') = (\tau[u] \circledast V) *_n (\tau[u'] \circledast V') = u \cdot V *_n u' \cdot V'.$$

The second identity is obtained as follows, using compatibility of τ with units and the previous lemma:

$$1_s^{m+1} \cdot 1_T^m = \tau\big[1_s^{m+1}\big] \circledast 1_T^m = \tau\, 1_{[s]}^m \circledast 1_T^m = 1_{\tau[s]}^m \circledast 1_T^m = 1_{\tau[s] \circledast T}^m = 1_{s \cdot T}^m. \qquad \triangleleft$$

Now we assume that $m > n > p$.

Lemma 43. *(interchange) The following identities hold for any m-cylinders $U \rhd_n U'$ and $V \rhd_n V'$ such that $U \rhd_p V$ (so that $U' \rhd_p V'$), for any n-cylinders $S \rhd_p T$, and for any p-cylinder R:*

$$(U *_n U') *_p (V *_n V') = (U *_p V) *_n (U' *_p V'), \qquad 1_S^m *_p 1_T^m = 1_{S *_p T}^m, \qquad 1_{1_R^n}^m = 1_R^m.$$

Proof. We proceed by induction on p.
If $p = 0$, the first identity is obtained as follows (with $U : x \curvearrowright y$, $U' : x' \curvearrowright y'$, $V : z \curvearrowright t$ and $V' : z' \curvearrowright t'$):

$$\begin{aligned}
[(U *_n U') *_0 (V *_n V')] &= (x *_n x') \cdot [V *_n V'] * [U *_n U'] \cdot (t *_n t') && \text{(definition of } *_0\text{)} \\
&= (x *_n x') \cdot ([V] *_{n-1} [V']) * ([U] *_{n-1} [U']) \cdot (t *_n t') && \text{(definition of } *_n\text{)} \\
&= (x \cdot [V] *_{n-1} x' \cdot [V']) * ([U] \cdot t *_{n-1} [U'] \cdot t') && \text{(compatibility of } \cdot \text{ with } *_{n-1}\text{)} \\
&= (x \cdot [V] * [U] \cdot t) *_{n-1} (x' \cdot [V'] * [U'] \cdot t') && \text{(compatibility of } * \text{ with } *_{n-1}\text{)} \\
&= [U *_0 V] *_{n-1} [U' *_0 V'] && \text{(definition of } *_0\text{)} \\
&= [(U *_0 V) *_n (U' *_0 V')]. && \text{(definition of } *_n\text{)}
\end{aligned}$$

The second identity is obtained as follows (with $S : x \curvearrowright x'$ and $T : y \curvearrowright y'$), using compatibility of action and concatenation with units:

$$\begin{gathered}
[1_S^m *_0 1_T^m] = 1_x^m \cdot [1_T^m] * [1_S^m] \cdot 1_{y'}^m = 1_x^m \cdot 1_{[T]}^{m-1} * 1_{[S]}^{m-1} \cdot 1_{y'}^m = \\
1_{x \cdot [T]}^{m-1} * 1_{[S] \cdot y'}^{m-1} = 1_{x \cdot [T] * [S] \cdot y'}^{m-1} = 1_{[S *_0 T]}^{m-1} = \left[1_{S *_0 T}^m\right].
\end{gathered}$$

The third identity is obtained as follows, using compatibility of τ with units:

$$\left[1_{1_R^n}^m\right] = 1_{[1_R^n]}^{m-1} = 1_{\tau\left[1_{R^\natural}^n\right]}^{m-1} = \tau\, 1_{\left[1_{R^\natural}^n\right]}^{m-1} = \tau\left[1_{1_{R^\natural}^n}^m\right] = \tau[1_{R^\natural}^m] = [1_R^m].$$

If $p > 0$, we apply the induction hypothesis. ◁

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